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THE LAGUERRE TRANSFORM

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THE LAGUERRE TRANSFORM

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ABSTRACT

A novel transform is presented which maps continuum functions (such as probability distributions) into discrete sequences and permits rapid numerical calculation of convolutions, multiple convolutions, and Neumann expansions for Volterra integral equations. The transform is based on the Laguerre polynomials, associated Laguerre functions, and their convolution properties.

Part 1 of this paper deals with functions having support only on $[0,\infty)$. The resulting unilateral Laguerre transform finds applications in convolution of such functions, inversion of Laplace transform, and in solution to renewal and related Volterra integral equations.

Part 2 of this paper deals with functions having support on $(-\infty,\infty)$ via a bilateral Laguerre transform which is an extension of the unilateral transform. Applications of this technique include convolution of such functions and analysis of the Lindley process.

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PART 1

LAGUERRE TRANSFORMATION AS A TOOL FOR THE NUMERICAL SOLUTION OF INTEGRAL EQUATIONS OF CONVOLUTION TYPE

by

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Table of Symbols Used

Symbols	Page Introduced
$a^{(k)}(x)$	1 (k-th fold convolution)
	24 (k-th derivative)
$L_2(0,\infty)$ $(\mathcal{L}_2)$	6
$\ell_n(x)$	6, Appendix A
$L_n(x)$	6, Appendix A
$f_n^{\dagger}$ $(f_n^{L\dagger})$	6
$T_f^{\dagger}(u)$ $(T_f^{L\dagger}(u))$	8, 19
$f_n^{\#}$ $(f_n^{L\#})$	8
$T_f^{\#}(u)$ $(T_f^{L\#}(u))$	8
$\leftrightarrow$	11 (corresponds to)
g.f.	12 (generating function)
$e_m(\tau)$	15
$\mathcal{E}$ (set)	15
a_m^*	16
$T_a^{E\dagger}(w)$	16
$T_a^{E\#}(w)$	16
$Lg(t)$	28 (differential operator)
$L[\cdot]$	55 (Laplace Transform)
ζ_k	33
$\alpha(s)$, $\beta(s)$, $\gamma(s)$	(Laplace transform of a(t), b(t) c(t))

LAGUERRE TRANSFORMATION AS A TOOL FOR THE NUMERICAL SOLUTION OF INTEGRAL EQUATIONS OF CONVOLUTION TYPE

J. Keilson and W. R. Nunn

ABSTRACT

A novel transform is presented which maps continuum functions (such as probability densities) into discrete sequences and permits rapid numerical calculation of convolutions, multiple convolutions and Neumann expansions for Volterra integral equations. The transform is based on the Laguerre polynomials, associated Laguerre functions and their simple convolution properties. A second transform employs Erlang functions as elements of the basis. The limitations and advantages of the two transforms are discussed. Numerical inversion of Laplace transforms relates simply to the Erlang transform. The deconvolution of two functions, i.e., the solution of $a(t) = x(t)*b(t)$ may also be obtained quickly in this way.

INTRODUCTION

One often encounters in applied studies integral equations [16] either of form

$$\int_0^x a(x - x')f(x')dx' = b(x) \tag{1}$$

or of the form

$$f(x) - \int_0^x a(x - x')f(x')dx' = b(x) \tag{2}$$

where $a(x)$ and $b(x)$ are specified functions and $f(x)$ is to be found. Equations (1) and (2) are said to be Volterra integral equations of convolution type of the first and second kind respectively. The Neumann series solution of (2) has the form [19]

$$f(x) = b(x) + b(x) * \sum_1^{\infty} a^{(k)}(x) \tag{3}$$

where the asterisk denotes convolution and $a^{(k)}(x)$ is the k-fold convolution of $a(x)$ with itself.

The entity $\sum_0^{\infty} a^{(k)}(x)$ and matrix variants associated with systems of integral equations of convolution type arise in operations research [6], engineering [7], and biological studies [10].

Sometimes, differential-integral equations give rise to expressions such as

$$s(\tau) = \sum_{k=0}^{\infty} e^{-\lambda\tau} \frac{(\lambda\tau)^k}{(k+1)!} a^{(k+1)}(\tau) \tag{4}$$

which describes the busy-period density for certain $M|G|1$ queueing systems [18].

In easy cases the integral equations may be solved analytically via Laplace transformation, and full answers may be obtained when the Laplace transforms are invertible. More often than not, such transforms cannot be inverted and expressions such as (4) are of limited value when they cannot be evaluated explictly. The Laguerre transformation techniques developed in this paper may then be of value.

The deconvolution problem of finding $f(x)$ from (1) when $a(x)$ and $b(x)$ are known numerically, say, is particularly troublesome, and start-up difficulties described below may make conventional numerical procedures useless.

The Laguerre transform techniques described map continuum functions into sequences, and map the continuum convolution operation into lattice convolution of these sequences. Such discrete convolutions are well matched to modern computer competence, and the inversion mapping back to the continuum is direct.

Laguerre transformation has been developed as a tool for the solution of differential equations [12]. The applications of interest here are quite different and new tools have been needed to convert the underlying simple idea into a flexible working procedure adapted to computer requirements.

The first section introduces the Laguerre transform $T: f(\tau) \to (f_n^{L\dagger})_0^\infty$ in a form convenient for our needs. One has

$$f(\tau) = \sum_{n=0}^{\infty} f_n^{L\dagger} \ell_n(\tau) \tag{5}$$

for any square-integrable function $f(\tau)$ on $(0,\infty)$, where $\ell_n(\tau) = L_n(\tau)e^{-\tau/2}$ are the classical orthonormal Laguerre functions and $L_n(\tau)$ are the Laguerre polynomials. The notation of Abramowitz and Stegun [1] is employed throughout. Orthonormality provides the inverse transformation

$$f_n^{L\dagger} = \int_0^\infty f(\tau)\ell_n(\tau)d\tau \ . \tag{6}$$

Let $T_f^{L\dagger}(u) = \sum_0^\infty f_n^{L\dagger}u^n$ be the generating function of $f_n^{L\dagger}$. Then one has, as shown in Section 1,

$$T_f^{L\dagger}(u) = \frac{1}{1-u} \ \phi\left(\frac{1}{2} \ \frac{1+u}{1-u}\right) \tag{7}$$

where $\phi(s)$ is the Laplace transform of $f(\tau)$. This relationship permits evaluation of $f_n^{L\dagger}$ for many important $f(\tau)$.

Section 2 provides simple examples of the transform, and Section 3 discusses the structure of $T_f^{L\dagger}(u)$ in the complex u-plane. Such insight into structure in the complex plane is crucial to many of our algorithms and theorems.

It is often desirable to work with an expansion of the form

$$f(\tau) = \sum_0^\infty f_n^{E\dagger} \frac{\tau^n}{n!} e^{-\tau/2} \tag{8}$$

whose basis functions have convolution properties of comparable simplicity to the Laguerre functions. The mapping $T: f(\tau) \to (f_n^{E\dagger})_0^\infty$ will be called an Erlang transform. The set of functions so representable is more limited than that for the Laguerre basis. They must be integral functions of τ of order at most one, so that $f(\tau) = e^{-\tau^2}$ for example is excluded. The nature of the Erlang transform and its relationship to the Laguerre transform is described in Section 4.

The square integrability requirement on $f(\tau)$ for the Laguerre transform is not an intrinsic limitation. An exponential transformation is described in Section 5 which avoids such difficulties.

Even though square integrability suffices in principle for the Laguerre transform, practicability of the method requires that the coefficients $\left(f_n^{L\dagger}\right)_0^\infty$ fall off quickly with n so that computer time is not excessive. In Section 6, rapidity of disappearance of $(f_n^{L\dagger})$ with n is related to the smoothness and concentration of $f(\tau)$. In particular, it is shown that "rapidly decreasing" $f(\tau)$ for which $\tau^q(d/d\tau)^p f(\tau) \to 0$ as $\tau \to \infty$ for all non-negative q,p are associated with rapidly decreasing $(f_n^{L\dagger})$ for which $n^K f_n^{L\dagger} \to 0$, $n \to \infty$ for all positive K.

Algorithms for the calculation of the Laguerre coefficients are presented in Section 7. Section 8 is devoted to a discussion of the deconvolution problem.

A variety of numerical examples of the method are treated in Section 9, and the implementation of the procedure is discussed.

Section 10 describes interpolation methods and problems when the known functions are known only numerically.

A final section deals with possible generalizations of the method to special families of functions.

1. The Laguerre transform.

Let $f(x)$ be any function in $L_2(o,\infty)$, i.e., any square integrable measurable function on (o,∞). Then $f(x)$ may be expanded in terms of the Laguerre functions

$$\ell_n(x) = e^{-\frac{1}{2}x} \; L_n(x), \tag{1}$$

where $L_n(x)$ are the Laguerre polynomials having the Rodrigues formula

$$L_n(x) = \frac{1}{n!} \; e^x \frac{d^n}{dx^n} \; \{x^n e^{-x}\} \; . \tag{2}$$

Classical properties of the Laguerre polynomials and functions are given in an appendix. The Laguerre functions $\ell_n(x)$ are orthonormal on $(0,\infty)$, i.e.,

$$\int_o^\infty \ell_m(x)\ell_n(x)dx = \left\{\begin{matrix} 0, & m\neq n \\ 1, & m=n \end{matrix}\right\} , \tag{3}$$

and provide a complete basis for L_2 in the associated Hilbert space metric. For $f(x) \in L_2$, one has the Fourier-Laguerre expansion

$$f(x) = \sum_o^\infty f_n^+ \ell_n(x), \tag{4}$$

where, from (3)

$$f_n^{\dagger} = \int_0^{\infty} f(x)\ \ell_n(x)\ dx\ . \tag{5}$$

Equation (5) describes a mapping of the function $f(x)$ on the continuum $(0,\infty)$ into a sequence $\left(f_n^{\dagger}\right)_0^{\infty}$, i.e. into a function on the non-negative integers. This sequence will be called the Laguerre transform of $f(\tau)$. It should be noted that "Laguerre transform" is often used to denote the mapping $\int_0^{\infty} F(x)e^{-x} L_n(x)dx = \int_0^{\infty} F(x)e^{-x/2}\ \ell_n(x)dx$. This mapping has been employed as a tool for the study of differential equations [11].

From (5) one has the Parseval relation

$$\sum_0^{\infty} \left(f_n^{\dagger}\right)^2 = \int_0^{\infty} f^2(x)dx\ . \tag{6}$$

Two useful results from analysis are needed. The Laguerre functions have the generating function

$$\sum_0^{\infty} \ell_n(x)\ u^n = (1-u)^{-1}\exp\{-\tfrac{1}{2}\ x\ (1+u)(1-u)^{-1}\}\ ;\ 0 \le u < 1, \tag{7}$$

and Laplace transform

$$\lambda_n(s) = \int_o^\infty e^{-sx}\ell_n(x)\,dx = \frac{1}{s+\frac{1}{2}} \left(\frac{s-\frac{1}{2}}{s+\frac{1}{2}}\right)^n \tag{8}$$

From (8) one may verify the basic convolution property of the Laguerre functions

$$\ell_n(x)*\ell_m(x) = \int_o^x \ell_n(x-y)\ell_m(y)\,dy \tag{9}$$

$$= \ell_o(x)*\ell_{m+n}(x) = \ell_{m+n}(x) - \ell_{m+n+1}(x)$$

This convolution property underlies the Laguerre transform and its utility. For any $f(x)$ in L_2 with Laguerre transform $\left(f_n^\dagger\right)_o^\infty$ we define a pair of related generating functions $T_f^\dagger(u)$ and $T_f^\#(u)$, and the transform sequence $(f_n^\#)_o^\infty$ by

$$T_f^\dagger(u) = \sum_o^\infty f_n^\dagger u^n\,, \quad 0 \le |u| < 1\,; \tag{10a}$$

$$T_f^\#(u) = \sum_o^\infty f_n^\# u^n = (1-u) \sum_o^\infty f_n^\dagger u^n, \; o \le |u| < 1\,. \tag{10b}$$

The reason for the factor $(1-u)$ in (10) will soon be clear. From (6) one has $f_n^\dagger \to o$, as $n\to\infty$, so that $T_f^\#(u)$ is regular in the

interior of the unit circle $|u| < 1$ in the complex u-plane. From (10b) one has

$$f_n^{\#} = f_n^{\dagger} - f_{n-1}^{\dagger}\ ,\ n \geq 1\ ;\ f_o^{\#} = f_o^{\dagger} \tag{11}$$

so that the sequence $(f_n^{\#})$ is square summable as well. Employment of (5) in (10) gives

$$T_f^{\#}(u) = (1-u) \sum_{o}^{\infty} u^n \int_o^{\infty} f(x)\ell_n(x)\, dx\ ,\ o \leq u < 1.$$

When $f(x)$ is integrable on (o,∞), since $|\ell_n(x)| \leq 1$ the order of summation and integration may be interchanged. Hence from (7) one has

$$T_f^{\#}(u) = \int_o^{\infty} f(x) \exp\{-\tfrac{1}{2}\, x\, (1+u)(1-u)^{-1}\}\, dx$$

i.e.,

$$T_f^{\#}(u) = \phi\, (\tfrac{1}{2}\, (1+u)(1-u)^{-1})\ ,\ o \leq u < 1\ , \tag{12}$$

where $\phi(s) = \int_o^{\infty} e^{-sx} f(x)dx$ is the Laplace transform of $f(x)$.

Let $f*g = \int_o^{x} f(x-y)g(y)dy$. Since $\phi_{f*g}(s) = \phi_f(s)\ \phi_g(s)$, one has from (12)

$$T^{\#}_{f*g}(u) = T^{\#}_{f}(u)\ T^{\#}_{g}(u) , \tag{13}$$

for functions f,g that are both integrable and square integrable on (o,∞). It follows from (13) moreover that

$$(f*g)^{\#}_{n} = \sum_{o}^{n} f^{\#}_{n-m}\ g^{\#}_{m} . \tag{14}$$

The value of the Laguerre transform is now clear. The transformation via (9) and (11) maps functions $f(x)$, $g(x)$ into sequences $(f^{\#}_{n})$, $(g^{\#}_{n})$, and their continuum convolution $f(x)*g(x)$ is mapped into a lattice convolution, and thence back onto the continuum via

$f^{\dagger}_{n} = \sum_{o}^{n} f^{\#}_{m}$ and the representation (4). The advantage of the procedure for numerical convolution of two functions known only numerically is moot. But its value for iterative convolutions and multiple convolutions of a function $f(x)$ with itself, and weighted sums of such multiple convolutions (i.e. (0.4)) via machine computation is apparent. Some of the applications of such computation in statistics and applied probability will be presented subsequently. The transform also has analytical value, as we will also attempt to show.

2. Some Elementary Laguerre Transforms

Exponential and Erlang Functions

The simplest and most basic set of Laguerre families is that for the exponential functions $f(t) = e^{-\theta\tau}$. Here, since $\phi(s) = (\theta+s)^{-1}$, we have from (1.12)

$$T_f^{\#}(u) = \frac{(1-u)}{(\theta+\tfrac{1}{2}) - (\theta-\tfrac{1}{2})u} \tag{1}$$

and hence

$$f_n^{\dagger} = \frac{1}{\theta+\tfrac{1}{2}} \left(\frac{\theta-\tfrac{1}{2}}{\theta+\tfrac{1}{2}}\right)^n \tag{2}$$

The Erlang densities for scale parameter θ are correspondingly simple, since they arise from convolutions of the exponential densities. They are given in the table of transforms.

Laguerre Functions

A related set of transforms of special interest is that for the Laguerre functions. Here from (1.10) we see that

$$\ell_n(\tau) \leftrightarrow T_f^{\#}(u) = (1-u)u^n \tag{3}$$

in particular

$$\ell_o(\tau) = e^{-\tau/2} \leftrightarrow T_f^{\#}(u) = (1-u) \tag{4}$$

so that for its N-fold convolution

$$\left[e^{-\tau/2}\right]^{(N)} = \frac{\tau^{N-1}}{(N-1)!}\, e^{-\tau/2} \leftrightarrow T_f^{\#}(u) = (1-u)^N \tag{5}$$

For this family of functions and finite mixtures thereof, the transform g.f.'s $T_f^{\#}(u)$ are polynomials and the transform sequences $f_n^{\#}$ terminate. The value $\theta=\frac{1}{2}$ thus provides a natural scale for computation in which convergence is most efficient. This will be discussed further subsequently.

3. Structure of $T_f^{L+}(u)$ in the complex plane.

When $f(t) \varepsilon L_2(o,\infty)$, Equation (1.6) implies that

$\sum_o^\infty (f_n^\dagger)^2 < \infty$, so that $f_n^\dagger \to 0$, as $n \to \infty$, and $\sum_o^\infty f_n^\dagger u^n = T_f^\dagger(u)$ is regular inside the unit circle, i.e. for $0 \leq |u| < 1$. Moreover $f_n^\# \to 0$ (and is square summable) so that $T_f^\#(u)$ is regular for $0 \leq |u| < 1$, as may also be seen from $T_f^\#(u) = (1-u)T_f^\dagger(u)$. It may be noted that $|u| = 1$ maps into the line $Re(s) = 0$, i.e. the imaginary axis in the s-plane, and that the point $u = 1$ corresponds to $s = \infty$.

For the representation $f(\tau) = \Sigma f_n^\dagger \ell_n(\tau)$, where $|\ell_n(\tau)| \leq 1$, speed of convergence is promoted by a geometric decay rate for $f_n^\dagger$, and hence by regularity of $T_f^\dagger(u)$, $T_f^\#(u)$ in a region containing the unit circle in its interior. This requires (cf. (1.12)) that $\phi(s)$ be regular at $s = \infty$ and have a negative abscissa of convergence [20]. Such behavior in the s-plane of $\phi(s)$ also assures regularity of $T_f^\#(u)$ and $T_f^\dagger(u)$ in and on the unit circle. The situation is summarized by the following theorem.

<u>Th`m</u> If $f(\tau)\, e^{\gamma\tau} \varepsilon L_1\ (o,\infty)$ for some $\gamma > o$ and if $\phi(w^{-1}) = \int_o^\infty f(\tau) e^{-\tau/w}\, d\tau$ is regular at $w=o$, and vanishes there,

then $T_f^{\#}(u) = \phi(\frac{1}{2}(1+u)(1-u)^{-1})$ and $T_f^{+}(u)$ are regular in the circle $o \leq |u| \leq c$, some $c > 1$.

Some examples may be of interest

<u>Ex. 1</u> $f(\tau) = \tau e^{-\beta\tau}$; $\phi(s) = \dfrac{1}{(s+\beta)^2}$;

$$T_f^{\#}(u) = \left(\frac{1}{2}\,\frac{1+u}{1-u} + \beta\right)^{-2}$$

which has a pole of order 2 at $(2\beta+1)/(2\beta-1)$, and is regular elsewhere. $T_f^{\#}(u)$ has radius of convergence $R_c > 1$.

<u>Ex. 2</u> $f(\tau) = \sqrt{\tau}\ e^{-a\tau/2}$; $\phi(s) = \frac{1}{2}\sqrt{\pi}\ (s+\frac{a}{2})^{-3/2}$;

$$T_f^{\#}(u) = \sqrt{2\pi}\ \left\{\frac{1+u}{1-u} + a\right\}^{-3/2}$$

and this has a branch point at $u = 1$. So $R_c = 1$.

Note that $f(\tau)$ is real analytic on (o,∞) and falls off rapidly in example 2, but that $R_c = 1$ for $T_f^{\#}(u)$. We will return to such behavior in Section 6.

4. The Erlang Family of Functions, and the Erlang Transform.

The class of functions $L_2(o,\infty)$ is too broad for some numerical work. The functions one deals with are often highly smooth and such smoothness permits representation by sequences (a_n) which fall off quickly as $n \to \infty$. An alternative to the Laguerre building blocks may then be available as we next describe.

Definition. Let $e_m(\tau) = \frac{\tau^m}{m!} e^{-\tau/2}$, and let $\mathcal{E}$ be the family of functions (entire functions as we will see)

$$\mathcal{E} = \{a(\tau): \quad a(\tau) = \sum_0^\infty a_m^* e_m(\tau)\} \tag{1a}$$

where

$$\sum_0^\infty \left| a_m^* \right| \beta^m < \infty, \quad \text{some} \quad \beta > 0. \tag{1b}$$

This family of functions will be called the Erlang family.

As will be seen condition (1b) permits use of generating functions and discrete convolution related thereto. We note that $\mathcal{E}$ is a linear space. It follows from (1a), and (1b) that $a(\tau)e^{\tau/2} = \sum_0^\infty (a_m^*/m!)\tau^m$ is a power series with infinite radius of convergence and is therefore an entire function. Hence $a(\tau)$ is also entire.

<u>Definition</u>. Let $a(\tau) \in E$. The sequence $(a_m^*)_0^\infty$ will be called the <u>Erlang transform</u>† of $a(\tau)$.

Since $a(\tau)e^{\tau/2} = \sum_0^\infty a_m^* \tau^m/m!$ is entire and hence regular at $\tau = 0$, it follows from the Taylor expansion that

$$a_m^* = (\frac{d}{d\tau})^m \left[a(\tau)e^{\tau/2}\right]_{\tau=0} = m! \left[\left(\frac{a^{(k)}(o)}{k!}\right) * \left(\frac{2^{-k}}{k!}\right)\right]_m$$

i.e. that

$$a_n^* = \sum_{m=0}^{n} \left\{\frac{1}{2^m}\binom{n}{m} a^{(n-m)}(o)\right\} . \tag{2}$$

Equation (2) provides the Erlang expansion of any function $a(\tau)$ in the Erlang family from its Taylor expansion about $\tau = 0$.

The Laplace transform of $a(\tau)$ is given from (1) by

$$\alpha(s) = \sum_0^\infty a_m^* \, (s+\tfrac{1}{2})^{-m-1} \tag{3}$$

and this series will converge absolutely when the real part of s is sufficiently large. Let

$$T_a^{E\dagger}(w) = \sum_0^\infty a_m^* w^m; \; T_a^{E\#}(w) = w \sum_0^\infty a_m^* w^m \tag{4}$$

Then

$$T_a^{E\#}\left(\frac{1}{s+\frac{1}{2}}\right) = \alpha(s), \tag{5a}$$

†The transform might equally be called a Maclaurin transform. The name Erlang has been used because the Laplace transform structure of the Gamma densities and Erlang densities suggested the method.

i.e.

$$T_a^{E\#}(w) = \alpha\left(\frac{2-w}{2w}\right) . \tag{5b}$$

Hence we again have, for $a(\tau), b(\tau) \in \mathcal{E}$

$$T_{a*b}^{E\#}(w) = T_a^{E\#}(w)\ T_b^{E\#}(w) \tag{6}$$

Again, as in Section 1, (6) permits study and evaluation of continuum convolution via lattice convolution.

The validity of the transform $T_a^{E\#}(w)$ requires that this be regular in w at $w = 0$, and this will be assured when $\alpha(s)$ is regular at ∞. Indeed such regularity provides an alternate characterization of members of the Erlang family, as we see next.

Theorem 4.1

$$a(\tau) \underset{\in}{\overset{\underline{A}}{}} \mathcal{E} \iff \overset{\underline{B}}{\text{The Laplace transform}}\ \alpha(s) \text{ of } a(t) \text{ is regular at infinity and vanishes there.}$$

Proof. We have seen that $A \Rightarrow B$. Suppose conversely that B is true. Then $\alpha(s)$ regular at ∞ implies $\alpha(s-\frac{1}{2})$ regular at ∞, i.e.

$$\alpha(s-\tfrac{1}{2}) = \int\{a(\tau)e^{\tau/2}\}\ e^{-s\tau}d\tau = \sum_{0}^{\infty} c_m(1/s)^{m+1}$$

where the power series on the right has a positive radius of convergence. But $\sum_{0}^{\infty} c_m\ (\frac{1}{s})^{m+1}$ is the Laplace transform of

$\sum_{0}^{\infty} (C_m/m!)t^m$ which must equal $a(\tau)e^{\tau/2}$ since Laplace transforms are one to one, and the conclusion follows.

Closure properties of the space E are also of interest, and follow at once from Theorem 4.1.

Theorem 4.2. Let $a_1(t)$, $a_2(t) \varepsilon E$. Then

(1) $\quad c_1 a_1(t) + c_2 a_2(t) \in E$;

(2) $\quad a_1(t)e^{\gamma\tau} \in E$, for all real γ ;

(3) $\quad a_1(t) * a_2(t) \in E$ (closure under convolution)

We observe that the Laguerre functions $\ell_n(\tau)$ and finite linear combinations thereof are all elements of E. But other elements of E need not be in $L_2(o,\infty)$ nor even go to zero as $t \to \infty$. Thus for $a(\tau) = \tau^2$, $\alpha(s) = 2s^{-3}$ which is regular at infinity. Similarly every polynomial in τ is in the space E.

It may be seen that the order of $a(\tau)$ is at most equal to 1. Hence E is a proper subset of the set of entire functions and $e^{-\tau^2}$ cannot be in E.

Note that the space E is not closed under limits. Thus $\alpha_N(s) = (1 + s/N)^{-N}$ converges to e^{-s}, i.e. a sequence of Laplace transforms regular at ∞ converges to one which is not.

A subspace of E of special interest is EL_2 the intersection of E and $L_2(o,\infty)$. If $a(\tau) \varepsilon EL_2$, it will have both an Erlang transform and a Laguerre transform. If the Laguerre related function $T_a^{\#}(u)$ of (1.10b) is designated by $T_a^{L\#}(u) = \alpha\left(\frac{1}{2}\ \frac{1+u}{1-u}\right)$ we have from 5b)

$$T_a^{E\#}(w) = T_a^{L\#}(1-w), \tag{7}$$

$$T_a^{L\#}(u) = T_a^{E\#}(1-u) \tag{8}$$

The module functions $\{e_n(\tau)\}$ and $\{\ell_n(\tau)\}$ of the two transforms are easily related. From (7),(8), since $e_n(\tau) = [\ell_o(\tau)]^{(n+1)}$ we have $T_{e_n}^{E\#}(w) = w^{n+1}$, and $T_{e_n}^{L\#}(u) = (1-u)^{n+1}$ so that

$$e_n(\tau) = \sum_o^n \theta_{nr}\ell_r(\tau) \tag{9}$$

where $\theta_{nr} = \binom{n}{r}(-1)^r$. Similarly

$$\ell_n(\tau) = \sum_o^n \theta_{nr}e_r(\tau) \tag{10}$$

That (θ_{mn}) is its own inverse is a classical result. (See for example, Riordan [15]. If one has $a(\tau) \varepsilon EL_2$, then

$$a(\tau) = \sum_{o}^{\infty} a_n^{\dagger}\ell_n(\tau) = \sum_{o}^{\infty} a_m^{*}e_m(\tau) \tag{11}$$

and the sequences $(a_n^{\dagger})_o^{\infty}$ and $(a_m^{*})_o^{\infty}$ are interrelated by

$$a_m^{*} = \sum_{n=m}^{\infty} a_n^{\dagger}\theta_{nm} \tag{12a}$$

$$a_n^{\dagger} = \sum_{m=n}^{\infty} a_m^{*}\theta_{mn} \tag{12b}$$

For any such function in EL_2 the Erlang transform coefficients a_m^{*} obtained from (2) provide a stepping stone to the Laguerre transform $(a_n^{\dagger})_o^{\infty}$ via (12b).

The Erlang transform method for evaluating convolutions is closely related to the method of phases employed by Neuts and others [13,14] for the numerical evaluation of functions arising in stochastic models. Indeed the method of phases provides an intuitive framework for understanding the Erlang transform method. The latter method is more general than the method of phases in several respects.

a) It is applicable to convolutions of functions of mixed sign, and hence to the response functions of electrical engineering arising in circuits and propagation, for example.

b) For positive functions no auxiliary Markov chains models are required. Indeed, certain simple probability density functions,

such as $Ke^{-a\tau}[1-\cos b\tau]$, cannot be related to chain models, as pointed out by D. R. Cox [4]. Such functions present no difficulty for the Erlang transform method.

c) The method of phases requires, in principle, integrable functions. A function such as $I_0(t)e^{-t}$, where $I_0(\tau)$ is the modified Bessel function, is unacceptable. The Erlang method may be employed even for functions which grow exponentially.

5. Extension of the Laguerre Transform to Non-Integrable Functions Via Exponential Transformation

Our applications of motivating interest center about the convolution operation $d(t) = a(t) * b(t)$. When $a(t)$ and $b(t)$ are not in $L_2(o,\infty)$, they do not have Laguerre transforms. We note however that when $a(t)$ and $b(t)$ are Laplace-transformable, one has $\delta(s) = \alpha(s)\beta(s)$ for s sufficiently large. Then $\alpha(s+\theta)\beta(s+\theta) = \delta(s+\theta)$, $\theta > 0$, i.e. $a_\theta(t) * b_\theta(t) = d_\theta(t)$, where $a_\theta(t) = a(t)e^{-\theta t}$, etc. When a_θ, $b_\theta \in L_2$, they may be convolved via Laguerre transform methods to give d_θ and one may then find $d(t)$ from $d(t) = d_\theta(t)e^{\theta t}$. Such exponential transformation extends thereby the scope of Laguerre transform methods. Note that the same procedure may be employed for the integral equations of convolution type of the introduction. For example the equation

$$h(t) = b(t) + a(t) * h(t)$$

becomes

$$h_\theta(t) = b_\theta(t) + a_\theta(t) * h_\theta(t)$$

under such transformation, and the Laguerre transform method may be used when $a_\theta, b_\theta \in L_2$ even though $a,b \notin L_2$.

6. The rate of convergence of the Laguerre coefficients

It is well known that a Fourier series expansion of a periodic function $f(t) = \sum a_n e^{int}$ has coefficients a_n which fall off rapidly with n when $f(t)$ is sufficiently smooth [5]. A similar behavior for the coefficients $f_n^{\dagger}$ of (1.4) and $f_n^{\#}$ of (1.11) is available and of key importance to the methods we desire.

We first observe that the square summability of $(f_n^{\dagger})$, i.e., $\sum_0^{\infty} (f_n^{\dagger})^2 < \infty$ implies that $(f_n^{\dagger})^2 \to 0$ as $n \to \infty$. Our methods require convolution of sequences $(f_n^{\dagger})$, $(g_n^{\dagger})$. As remarked in Section 3, for accuracy of calculation we would like the coefficients to fall off rapidly, and geometric decay of the coefficients would be desirable.

It may be noted that summability $\sum_0^{\infty} |f_n^{\dagger}| < \infty$ implies the continuity of $f(t)$ in (1.4). The argument is along classical lines. Since $|\ell_n(t)| \le 1$, one has $|\ell_n(t+h) - \ell_n(t)| \le 2$. Hence by dominated convergence

$$\lim_{h\to 0} f(t+h) - f(t) = \lim_{h\to o} \sum_0^{\infty} f_n^{\dagger} \{\ell_n(t+h) - \ell_n(t)\} = 0.$$

and $f(t)$ is continuous.

It then follows that when $f(t)$ is not continuous for all t in $(0,\infty)$, that $(f_m^{\dagger})$ is not summable. For example the

function $f(t) = 1$, $0 \leq t \leq 1$; $f(t) = 0$, $1 < t$, has coefficients which are square summable but not summable and hence rather slow in convergence to zero. Dominated convergence also leads to the conclusion that $f(\tau) = \sum f_n^\dagger \ell_n(\tau)$ goes to zero as $\tau \to \infty$ when $\sum |f_n^\dagger| < \infty$.

The argument can be generalized to the following theorem. Let $C_\downarrow^k(R_+)$ be the linear space of functions $f(\tau)$ with continuous k'th derivatives on (o,∞), such that $\lim\limits_{\tau\to\infty} f^{(k)}(\tau) = 0$.

<u>Theorem 2.1</u>. Let $f(\tau) = \sum\limits_0^\infty f_m^\dagger \ell_m(\tau)$. Then

$\sum\limits_{m=0}^{\infty} m^K |f_m^\dagger| < \infty \Rightarrow f(\tau) \in C_\downarrow^k(R_+)$, $k = 1,2,\ldots.K$. The proof is based on the following lemmas.

<u>Lemma 2.2</u> $|\ell_m^{(k)}(\tau)| \leq |\ell_m^{(k)}(0)|$

<u>Proof</u>. The standard identity $L_m'(\tau) = -\sum\limits_0^{m-1} L_r(\tau)$ implies

that $-\ell_m'(\tau) = \frac{1}{2}\ell_m(\tau) + \sum\limits_0^{m-1} \ell_r(\tau) = \sum\limits_{r=0}^{m} \theta_{mr}\ell_r(\tau)$ where $\theta_{mm} = \frac{1}{2}$,

$\theta_{mr} = 1$, $0 \leq r \leq m-1$. Hence, by induction $(-1)^k \ell_m^{(k)}(\tau) = \sum\limits_0^m (\underline{\Theta}^k)_{mr}\, \ell_r(\tau)$ where $\underline{\Theta} = (\theta_{mn})$ is the associated lower triangular matrix and $\underline{\Theta}^k$ is non-negative for all k. Hence $|\ell_m^{(k)}(\tau)| \leq \sum\limits_{r=0}^{m} (\underline{\Theta}^k)_{mr} = |\ell_m^{(k)}(0)|$.

<u>Lemma 2.3</u> $\binom{m}{k} \le (-1)^k \ell_m^{(k)}(0) \le \binom{m+k}{k}$

<u>Proof:</u> $\sum_0^\infty u^m \ell_m(\tau) = (1-u)^{-1} \exp\{-\tfrac{1}{2}\tau(1+u)/(1-u)\} \Rightarrow$

$$\sum_{m=0}^\infty u^m \ell_m^{(k)}(0) = (-\tfrac{1}{2})^k \frac{(1+u)^k}{(1-u)^{k+1}} = (-\tfrac{1}{2})^k \sum_{r=0}^k \binom{k}{r} u^r (1-u)^{-k-1}.$$

Hence $(-1)^k \ell_m^{(k)}(0) = \sum_{r=0}^k (\tfrac{1}{2})^k \binom{k}{r}\binom{m+k-r}{k} = \sum_{r=0}^k (\tfrac{1}{2})^k \binom{k}{r}\binom{m+r}{k}$. But

$\binom{m}{k} \le \binom{m+r}{k} \le \binom{m+k}{k}$, $0 \le r \le k$, and $\sum_0^k \binom{k}{r} = 2^k$, so that the lemma follows.

<u>Proof of Theorem 2.1</u>. Clearly* $\binom{m+k}{k} \le (k+1)m^k$, $m \ge 1$, so that $|\ell_m^{(k)}(\tau)|/m^k \le k+1$, and $|\ell_m^{(k)}(\tau)|/m^K \le k+1$, $1 \le k \le K$, $m \ge 1$. Consider $f(\tau) = \sum_1^\infty (f_m^+ m^K) \frac{\ell_m(\tau)}{m^K} + f_0^+ \ell_0(\tau)$

and $$\frac{f(\tau+h) - f(\tau)}{h} = \sum_{m=1}^\infty \frac{(f_m^+ m^K)}{h} \int_\tau^{\tau+h} \frac{\ell_m'(y)\,dy}{m^K} + f_0^+ \frac{\ell_0(\tau+h) - \ell_0(\tau)}{h}.$$

By dominated convergence, we have $f'(\tau) = \sum_0^\infty f_m^+ \ell_m'(\tau)$ and $f'(\tau+h) - f'(\tau) = \sum_0^\infty f_m^+ \left[\ell_m'(\tau+h) - \ell_m'(\tau)\right]$, so that $f'(\tau)$ is continuous and goes to zero as $\tau \to \infty$. The argument may then be repeated for all derivatives up to and including the K'th derivative.□

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* $\binom{m+k}{k} = \frac{(m+1)(m+2)\ldots.(m+k)}{1\cdot 2\ldots.k} \le m^k \frac{2\cdot 3\ldots.(k+1)}{1\cdot 2\ldots.k} = (k+1)m^k$.

A simple extension of the theorem will be of interest subsequently.

<u>Definition</u> $(a_n) \varepsilon S_K \iff \sum_0^\infty n^K |a_n| < \infty$

<u>Theorem 2.4a</u>. Let $f(\tau) = \sum_0^\infty f_n^\dagger \ell_n(\tau)$, and let $(f_n^\dagger) \varepsilon S_K$.

Let $\tau^p (\frac{d}{dt})^q f(\tau) = \sum_0^\infty f_n^{\dagger(p,q)} \ell_n(\tau)$; $\begin{matrix} p \geq 0, \ q \geq 0 \\ p + q \leq K. \\ K \geq 1 \end{matrix}$ Then

$(f_n^{\dagger(p,q)}) \varepsilon S_{K-(p+q)}$,

<u>Proof</u>. The theorem is proven by showing that differentiation lowers K by at most one, and multiplication by τ lowers K by at most one.

Thus let $g(\tau) = \sum g_n^\dagger \ell_n(\tau)$, with $\sum n^K |g_n^\dagger| < \infty$, $K \geq 1$.

Then $g(\tau) \varepsilon C_\dagger^K(R_+)$ and $h(\tau) = g'(\tau) = \sum_0^\infty g_n^\dagger \ell_n'(\tau) = \sum_0^\infty h_n^\dagger \ell_n(\tau)$

where $|h_n^\dagger| \leq \sum_{m=n}^\infty |g_m^\dagger|$, as in the proof of Lemma 2.2. Then

$$\sum_0^\infty n^{K-1} |h_n^\dagger| \leq \sum_{n=0}^\infty \sum_{m=n}^\infty n^{K-1} |g_m^\dagger| = \sum_{m=0}^\infty |g_m^\dagger| \sum_{n=0}^m n^{K-1} \leq C_K \sum_m |g_m^\dagger| m^K < \infty.$$

Similarly let $s(\tau) = \tau g(\tau) = \sum_{0}^{\infty} g_n^{\dagger} \tau \ell_n(\tau)$ with $(g_n^{\dagger}) \varepsilon S_K$, $K \geq 1$. Then $s(\tau) = \sum s_n^{\dagger} \ell_n(\tau) = \sum g_n^{\dagger} \{(2n+1)\ell_n(\tau) - (n+1)\ell_{n+1}(\tau) - n\ell_{n-1}(\tau)\}$, so that $(s_n^{\dagger})$ is summable. Then $s_n^{\dagger} = (2n+1)g_n^{\dagger} - -(n+1)g_{n+1}^{\dagger} - ng_{n-1}^{\dagger}$, and $\sum_{0}^{\infty} n^{K-1}|s_n^{\dagger}| < \infty$ as needed.

<u>Corollary 2.4b</u>. $\left.\begin{array}{l}\sum n^K |f_n^{\dagger}| < \infty \\ K>1,\ p,q\geq 0 \\ p+q \leq K\end{array}\right\} \Rightarrow \tau^p (\frac{d}{dt})^q f(\tau) \ \varepsilon\ C_{\downarrow}^{K-p-q}(R_+)$

Let us now see how smoothness in $f(t)$ induces rapidity of convergence in $(f_n^{\dagger})$. The Laguerre function $\ell_n(t)$ satisfies [1] the second order differential equation

$$t\,\ell_n''(t) + \ell_n'(t) + (n+\tfrac{1}{2}-\tfrac{1}{4}t)\ell_n(t) = 0 \qquad (1)$$

for $0 < t < \infty$. This may be written in the form

$$\begin{aligned}\left[\tfrac{1}{4}t - \frac{d}{dt}\,t\,\frac{d}{dt}\right] \ell_n(t) &= \left[\tfrac{1}{4}t - D\right] \ell_n(t) \qquad (2)\\ &= L\,\ell_n(t)\\ &= (n+\tfrac{1}{2})\,\ell_n(t)\end{aligned}$$

It is easily seen that the operator L is self-adjoint in the sense that

$$\int_0^\infty g(t)\ \Big[Lh(t)\Big]\ dt = \int_0^\infty \Big[Lg(t)\Big]\ h(t)\ \ dt \tag{3}$$

under simple conditions, (4a) and (4b) given below. Indeed one may employ the identity

$$\begin{aligned} g(t)\ Dh(t) - h(t)\ Dg(t) &= h(t)\ Lg(t) - g(t)\ Lh(t) \\ &= \frac{d}{dt}\left[g\ t\ \frac{dh}{dt} - h\ t\ \frac{dg}{dt}\right] \end{aligned}$$

and integrate by parts to obtain (3) provided that the function $g\ t\ \frac{dh}{dt} - h\ t\ \frac{dg}{dt}$ vanishes at $t = 0$ and ∞ and the integrands in (3) are in $L_1(0,\infty)$. Thus (3) is assured whenever $h(t) = \ell_n(t)$, $n \geq 1$, and

$$g^{(r)}(t)\ \text{ is continuous and bounded on }\ (0,\infty):\ \ r = 0,1,2 \tag{4a}$$

One further condition on $g(t)$ is needed as will be seen in a moment. This is the condition

$$Lg(t) = \tfrac{1}{4}tg(t) - tg''(t) - g'(t)\ \varepsilon\ L_2(0,\infty) \tag{4b}$$

From (2), (3), and (4a) we have for $g_n^\dagger = \int_0^\infty g(t)\ \ell_n(t)\ dt$,

$$g_n^{\dagger} = \left[\int_0^{\infty} g(t)\ L\ \ell_n(t)\ dt\right]\ (n+\tfrac{1}{2})^{-1} \tag{5}$$

$$= (n+\tfrac{1}{2})^{-1} \int_0^{\infty} \left[Lg(t)\right]\ \ell_n(t)\ dt$$

If $g(t)$ satisfies (4b), then $Lg(t) = \sum_0^{\infty} (Lg)_n^{\dagger}\ \ell_n(t)$ and $(Lg)_n^{\dagger} = o(1)$. Hence from (5) $g_n^{\dagger} = (n+\tfrac{1}{2})^{-1} o(1) = o(n^{-1})$.

The operator $L^k = (\tfrac{1}{4}t - D)^k$ is also self-adjoint under similar conditions on the boundary. From (2), one has

$$L^k\ \ell_n(t) = (n+\tfrac{1}{2})^k\ \ell_n(t) \tag{6}$$

If both sides of (6) are multiplied by $g(t)$, successive integrations by parts gives

$$(n+\tfrac{1}{2})^k\ g_n^{\dagger} = \int_0^{\infty} \ell_n(t)\ L^k g(t)\ dt \tag{7}$$

provided that $g(t)$ and $L^m g(t)$ satisfy conditions (4a,b) for $m = 1,2,\ldots,k-1$. These conditions will be assured if $g^{(r)}(t)$ is continuous and bounded on $(0,\infty)$ for $0 \leq r \leq 2k$. We again require that $L^k g(t) \in L_2(0,\infty)$, and then have, as above,

$$g_n^{\dagger} = o(n^{-k}),\ n \to \infty. \tag{8}$$

These conclusions are summarized in the following

Theorem 2.5. If

a) $f^{(r)}(t)$ is continuous and bounded on $(0,\infty)$, $r = 0,1,\ldots 2K$

b) $L^r f(t) \in L_2(0,\infty)$, $r = 0,1,\ldots,K$, (9)

then

$$f_n^{\dagger} = o(n^{-K}),\ n \to \infty \tag{10}$$

It may be noted that (9b) requires in general the existence of a 2K<u>th</u> moment for $f^2(t)$, i.e. that $\int_0^\infty t^{2K} f^2(t)\,dt < \infty$. Thus the coefficients $f_n^{\dagger}$ decay quickly to the extent to which $f(t)$ is smooth and localized about $t = 0$.

Of particular interest is the situation where $f(t)$ is real analytic on $(0,\infty)$ and falls off quickly enough at ∞ so that (10) can be true for all K. The class $C_\downarrow^\infty(R_1)$ of "rapidly decreasing functions" of interest to Fourier transforms (see Dym & McKean, [5], p. 87) is defined by: $f(x) \in C_\downarrow^\infty(R_1) \overset{\text{def}}{\iff} x^q \left(\frac{d}{dx}\right)^p f(x)$ is bounded on R_1 for all non-negative integers p,q. We may speak correspondingly of the class $C_\downarrow^\infty(R_+)$ of rapidly decreasing functions on $R_+ = (0,\infty)$ for which $x^q \left(\frac{d}{dx}\right)^p f(x)$ is bounded on R_+ for all non-negative integers p,q. Then as a corollary to our previous theorem and of the earlier Theorem 2.4 we have

Theorem 2.6. A) $f(t) \in C_{\downarrow}^{\infty}(R_+)$ if and only if B) $n^K f_n^\dagger \to 0$, $n \to \infty$ for all positive integers K.

Definition

When (B) holds, $(f_n^\dagger)_0^\infty$ is said to be a "rapidly decreasing sequence" [5].

Note that (B) implies that $n^K \Delta f_n^\dagger = n^K(f_{n+1}^\dagger - f_n^\dagger) \to 0$, $n \to \infty$ and indeed that $n^q D^p f_n^\dagger \to 0$, $n \to \infty$, where D is any forward, central or backward difference operator. The name rapidly decreasing sequence is therefore appropriate. Such functions as $n^K \theta^n$ and $n^K \theta^{n^2}$ are rapidly decreasing, for $0 \le \theta < 1$.

It is important to note that smoothness of $f(t)$ alone does not assure that $n^K f_n^\dagger \to 0$ for all K. As an example, let $f(t) = 1/(1+t) \in L_2$. Then $f^{(p)}(t) \to 0$, $t \to \infty$. We have

$$f_n^\dagger = \int_0^\infty \frac{1}{1+t}\,\ell_n(t)\,dt$$

$$= \int_0^\infty \int_0^\infty e^{-\theta(1+t)}\,\ell_n(t)\,d\theta\,dt$$

$$= \int_0^\infty \frac{1}{\theta+\frac{1}{2}} \left[\frac{\theta-\frac{1}{2}}{\theta+\frac{1}{2}}\right]^n e^{-\theta}\,d\theta$$

$$= e^{\frac{1}{2}} \int_{\frac{1}{2}}^\infty u^{-1}(1-u^{-1})^n e^{-u}\,du$$

$$= e^{\frac{1}{2}} \int_0^2 \{(1 - u)^n u^{-1} e^{-1/u}\} \, du$$

$$= I_n$$

For $n = 2k$, $I_{2k} = \{\int_0^1 \{ \quad \} \, du + \int_1^2 \{ \quad \} \, du\} e^{\frac{1}{2}}$

$$> e^{\frac{1}{2}} \int_1^2 (1 - u)^{2k} u^{-1} e^{-1/u} \, du$$

For $1 \le u \le 2$, $\left[e^{\frac{1}{2}} e^{-1/u} u^{-1}\right] \ge \frac{1}{2}$, hence $I_{2k} > \frac{1}{2} \int_0^1 x^{2k} dx = \frac{1}{2}(2k+1)^{-1}$

Hence $f_n^{\dagger}$ is not rapidly decreasing.

7. Calculation of the Laguerre Coefficients via Taylor Series.

We have seen that when $a(\tau) \in L_2$, $T_a^{L\#}(u) = \alpha(\frac{1}{2}\ \frac{1+u}{1-u})$ is regular about $u = 0$. There are two different Taylor series of value in calculating the Laguerre coefficients. The first valid in principle for all $a(\tau)$ in $L_2(R_+)$ proceeds as follows: We may write

$$\alpha(\frac{1}{2}\ \frac{1+u}{1-u}) = \alpha(\frac{1}{2} + \frac{u}{1-u}) = \int_0^\infty a(\tau)e^{-\tau/2}\ e^{-\frac{1}{2}\tau\{u/1-u)\}}d\tau$$

The function $\delta(u) = u/(1-u)$ is regular at $u = 0$ and $\alpha(\frac{1}{2} + \delta)$ is regular at $\delta = 0$. Thus

$$\alpha(\frac{1}{2} + \frac{u}{1-u}) = \sum_{k=0}^{\infty} \left\{\int_0^\infty a(\tau)e^{-\tau/2}\frac{(-\tau/2)^k}{k!}d\tau\right\} \frac{u^k}{(1-u)^k} \tag{7.1}$$

$$= \sum_{k=0}^{\infty} \zeta_k (\frac{u}{1-u})^k$$

But

$$(\frac{u}{1-u})^k = \sum_{m=k}^{\infty} \binom{m-1}{k-1} u^m \tag{7.2}$$

Hence

$$a_n^{L\#} = \sum_{k=1}^{n} \zeta_k \binom{n-1}{k-1} \tag{7.3}$$

and similarly, since

$$\frac{u^k}{(1-u)^{k+1}} = \sum_{m=k}^{\infty} \binom{m}{k} u^m \tag{7.4}$$

$$a_n^{L\dagger} = \sum_{k=0}^{n} \zeta_k \binom{n}{k} \tag{7.5}$$

Thus one has

Theorem 7.1. If $a(\tau) \in L_2(R_+)$, its Laguerre coefficients are given by (7.3) and (7.5).

Example: $a(\tau) = t^m e^{-\beta\tau}$, $\beta > 0$ has $\zeta_k = (\frac{-1}{2})^k \int_0^\infty \tau^{m+k} e^{-(\beta+\frac{1}{2})\tau} d\tau$

$$= (m+k)!\,(-\tfrac{1}{2})^k \,(\beta+\tfrac{1}{2})^{-m-k-1} \qquad \text{and}$$

$$a_n^{L\dagger} = \sum_{k=0}^{n} (-\tfrac{1}{2})^k (\beta+\tfrac{1}{2})^{-m-k-1} \binom{n}{k} (m+k)! \quad .$$

Note that Theorem 7.1 is useful when the ζ_k coefficients are available. In some circumstances a Taylor expansion of $\alpha(s)$ about $s = 0$ is more convenient because the Taylor coefficients there are available. Thus if

$$\alpha(s) = \int_0^\infty a(\tau) e^{-s\tau} d\tau$$

is regular for $0 \leq |s| < A$, $A > \frac{1}{2}$, the point $s = \frac{1}{2}$ lies inside the circle of convergence of the Taylor's series of $\alpha(s)$ about $s = 0$. If we write $\alpha(s) = \sum_0^\infty \alpha_n s^n$, then

$$\alpha\left(\frac{1}{2}\ \frac{1+u}{1-u}\right) = \alpha\left(\frac{1}{2} + \frac{u}{1-u}\right) = \sum_0^\infty \alpha_n \left(\frac{1}{2} + \frac{u}{1-u}\right)^n$$

$$= \sum_0^\infty \sum_0^\infty \alpha_n g_{nk} u^k = \sum_{k=0}^\infty \left\{ \sum_{n=0}^\infty \alpha_n g_{nk} \right\} u^k$$

It is then easy to show (see below) with the help of the Cauchy inequality that the series

$$a_k^{L\#} = \sum_{n=0}^\infty \alpha_n g_{nk}$$

provides an absolutely convergent representation of the Laguerre coefficients $(a_n^{L\#})$. The coefficients α_n are given by

$$\alpha_n = \frac{(-1)^n}{n!}\ \mu_n$$

where

$$\mu_n = \int_0^\infty a(\tau)\tau^n\, d\tau \ ,$$

and $$(\tfrac{1}{2} + \frac{u}{1-u})^n = \sum_{k=0}^{\infty} g_{nk} u^k , \quad 0 \leq |u| < 1.$$

Thus from the Binomial expansion and

$$\frac{u^r}{(1-u)^r} = \sum_{k=r}^{\infty} \binom{k-1}{k-r} u^k , \quad r \geq 1$$

we have $g_{n0} = (\frac{1}{2})^n$ and,

$$g_{nk} = \sum_{r=1}^{n} \binom{n}{r} (\tfrac{1}{2})^{n-r} \binom{k-1}{k-r} , \qquad k \geq 1$$

When $a(\tau) = e^{-\tau^2}$, $\tau > 0$, $A = \infty$, and the coefficients may be obtained in this way. Similarly, when $a(\tau) = 1$, $0 \leq \tau \leq 1$, the calculation is easy.

Theorem 7.2. Let $\alpha(s)$ be regular for $0 \leq |s| < A$, $A > \frac{1}{2}$. Let $\alpha(s) = \sum_{0}^{\infty} \alpha_n s^n$, and let $\left[\frac{1}{2} + u(1-u)^{-1}\right]^n = \sum_{0}^{\infty} g_{nk} u^k$. Then

$\sum_n \alpha_n g_{nk}$ is absolutely convergent.

Proof. By Cauchy's Inequality, $|\alpha_n| < M_1/R_1^n$ where $M_1 = \max_{|s|=R} |\alpha(s)|$, $\frac{1}{2} < R_1 < A$. Also

$$|g_{nk}| < \frac{\left\{\max\limits_{|u|=\delta} \left|\frac{1}{2} + \frac{u}{1-u}\right|\right\}^n}{\delta^k}, \quad 0 < \delta < 1$$

By taking δ sufficiently small, we may assure that $\max\limits_{\delta} |\tfrac{1}{2} + \frac{u}{1-u}| < R_2$, $R_2 < R_1$, and the theorem follows.

8. The deconvolution problem

Let $b(t) = a(t)*f(t)$, i.e.,

$$b(t) = \int_0^t f(t')a(t - t')dt' , \tag{1}$$

where $b(t)$ and $a(t)$ are known and $f(t)$ is sought. This will be called the deconvolution equation. When $b(t)$ and $a(t) \in L_2(0,\infty)$ and $b(t) = \Sigma\, b_n^{\dagger}\ell_n(t)$, $a(t) = \Sigma\, a_n^{\dagger}\ell_n(t)$, and $f(t) = \Sigma\, f_n^{\dagger}\ell_n(t)$ we have from Section 1,

$$(b_n^{\#}) = (a_n^{\#})*(f_n^{\#}) ,$$

i.e.,

$$b_n^{\#} = \sum_0^n a_m^{\#} f_{n-m}^{\#} \tag{2}$$

A formal solution of (1) via Laplace transformation has the form

$$\phi(s) = \beta(s)/\alpha(s) \tag{3}$$

and this may be solved simply by analysis in some easy cases.

We note from (3) and the uniqueness of Laplace transforms that (1) has a unique solution if it has any solution. A solution is usually guaranteed by physical considerations or by the mathematical origins of the problem.

Difficulty arises when $a(t)$ and $b(t)$ are known numerically from data, say. Numerical solution of (1) is most naturally attempted by discretizing the continuum, but there are often serious difficulties, as we will see.

The numerical solution of the lattice deconvolution equation (2) is in principle trivial when the coefficients $a_n^{\#}$ and $b_n^{\#}$ are known. For one has

$$b_0^{\#} = a_0^{\#} f_0^{\#} \quad ,$$

$$b_1^{\#} = a_0^{\#} f_1^{\#} + a_1^{\#} f_0^{\#} \quad ,\ldots,$$

so that, recursively,

$$f_n^{\#} = \left[b_n^{\#} - \sum_{m=0}^{n-1} f_m^{\#} a_{n-m}^{\#} \right] (a_0^{\#})^{-1} \tag{4}$$

provided $a_0^{\#} \neq 0$. Now

$$a_0^{\#} = a_0^{+} = \int_0^{\infty} a(t)\ell_0(t)dt = \int_0^{\infty} a(t)e^{-t/2}dt \tag{5}$$

and one may expect, in general, that $a_0^{\#} \neq 0$. One could have $a_0^{\#} = 0$ if $a(t)$ is in the space orthogonal to $\ell_0(t)$.

The deconvolution equation (1) has start-up difficulties in discretization approaches when

$$a(t) \sim Kt^m \ , \quad t \to 0+ \ , \ K > 0 \tag{6}$$

for some positive integer m. Note, however, that this does not cause trouble in the Laguerre procedure, as may be seen best from an example. If $a(t) = te^{-at}$, $\alpha(s) = (s+a)^{-2}$ and $T_a^{\#}(u) = \dfrac{1}{(a + \frac{1}{2}\frac{1+u}{1-u})^2}$ with $a_0^{\#} = T_a^{\#}(0) = \dfrac{1}{(a+\frac{1}{2})^2} \geq 0$. So that there are no start-up difficulties.

In those instances where $a_m^{\#} = 0$, $0 \leq m \leq M-1$, one will also have from $T_b^{\#}(u) = T_a^{\#}(u)T_f^{\#}(u)$ that $b_m^{\#} = 0$, $0 \leq m \leq M-1$. Then $T_b^{\#}(u)/T_a^{\#}(u)$ $u^{-M}T_b^{\#}(u)/u^{-M}T_a^{\#}(u)$, and the lattice deconvolution proceeds normally.

When $a(t)$ and $b(t)$ are integral functions with Laplace transforms regular at infinity of order at most one, it may be desirable to represent $a(t)$ and $b(t)$ in terms of the Erlang functions of Section 4. This procedure has the advantage that the Erlang building blocks are naturally ordered by their behavior at $t=0$ in that $t^m e^{-t/2}$ is smaller near zero than $t^n e^{-t/2}$ when $m>n$. Note that $\phi(s) = \beta(s)/\alpha(s)$ is regular at ∞ when $\alpha(s)$ and $\beta(s)$ are regular at ∞ and $b(t)/a(t)$ is bounded near zero. The condition is required for the existence of any solution by reasoning similar to that of the previous paragraph, since

$$\sum_0^\infty b_m s^{-(m+1)} / \sum_0^\infty a_m s^{-(m+1)} \sim (a_M/b_N) s^{-(M-N)},\ |s| \to \infty \text{ where } a_M,\ b_N$$

are the leading coefficients and $M>N$ is required. The solution $f(t)$ will then itself be in the Erlang family, and lattice deconvolution of the Erlang coefficients is viable.

9. Numerical Examples of the Method

We present some numerical examples of the transform methods for the calculation of multiple convolutions as well as for the solutions of Volterra equations (0.1), (0.2). All computations were done on a Burroughs 6750 computer in a time-sharing mode using APL as a programming language. The computations were in single precision (11 digits). Relevant formulas were usually coded in an "obvious", straightforward way, with no attempt made to optimize the subroutines for speed or accuracy. In spite of this, the results displayed in this section were typically gotten with CPU times measured in seconds, and with no evidence of numerical problems such as roundoff, truncation, underflow, etc.

The Laguerre functions were calculated using the recursion relation (see Appendix A)

$$\ell_{n+1}(t) = [2 - \frac{(t+1)}{n+1}]\ell_n(t) - [\frac{n}{n+1}]\ell_{n-1}(t)$$

with $\ell_0(t) = \exp(-vt)$ and $\ell_1(t) = (1-t)\exp(-vt)$. Similarly, $e_n(t) = (\frac{t}{n})e_{n-1}(t)$, with $e_0(t) = \exp(-vt)$. The use of the scale factor v is recommended for computational flexibility. When $v = \frac{1}{2}$, the above-mentioned Laguerre and Erlang functions are obtained. When $v = 0$, the Laguerre functions become the Laguerre polynomials and the Fourier-Erlang series reduces to the Taylor series. The use of the scale factor v is equivalent to the use of the exponential

transform of Section 5, for if $f_\theta(t) = f(t)e^{-\theta t}$, then $f_\theta(t) = \sum f_n^+ e^{-vt} L_n(t)$ for $v = \theta + \frac{1}{2}$. Hence the use of $v \neq \frac{1}{2}$ is justified providing $f(t)$ is Laplace transformable and $f_\theta(t) \in L_2$. Typically, the computer will give the same answer for any v in the vicinity of ½, and the user may choose a value of v empirically to get the best numerical accuracy.

The Laguerre and Erlang coefficients for exp(-at) are needed frequently in applications and are derived here for easy reference. For $f(t)\exp[(v-\frac{1}{2})t] \in L_2(0,\infty)$, (1.5) becomes

$$f_n^{L+}(v) = \int_0^\infty f(t)\exp\{(v-\tfrac{1}{2})t\}\ell_n(t)dt \tag{1}$$

(in which $\ell_n(t)$ is the classical Laguerre function) and for $f(t) = \exp(-at)$, we recognize the Laplace transform of $\ell_n(t)$ evaluated at $a + \frac{1}{2} - v$. From (1.8),

$$f_n^{L+}(v) = \frac{(a-v)^n}{(a+1-v)^{n+1}}\ , \quad n = 0,1,2,\ldots \quad . \tag{2a}$$

For the Erlang transform we have

$$f(t) = \exp(-at) = \sum f_n^{E+}(v)(t^n/n!)\exp(-vt) \quad ,$$

so that $\sum f_n^{E+}(v)(t^n/n!) = \exp(v-a)t$, and

$$f_n^{E+}(v) = (v-a)^n\ , \quad n = 0,1,2,\ldots \quad . \tag{2b}$$

These coefficients for the exponential function serve as useful building blocks for calculating coefficients of more involved functions. For example, if $f(t)$ is an Erlang density, its coefficients may be easily computed by recognizing $f(t)$ as an integral number of convolutions of an exponential density.

Example 1: Server Busy Period Density

This example illustrates the use of the transform methodology for the evaluation of multiple convolutions. Reference [3] illustrates the numerical effort that may be required to perform multiple convolutions by numerical methods. When the function being convolved has rapidly decreasing coefficients, our method yields a fast and accurate algorithm.

Consider a single server M/G/1 queueing system [18] having a Poisson stream of customers at rate λ and service time distribution which is absolutely continuous with density $a(t)$. The server busy period density $s(t)$ is well known [18] and is given by

$$s(t) = \sum_{0}^{\infty} \exp(-\lambda t) \frac{(\lambda t)^n}{(n+1)!} a^{(n+1)}(t) \quad . \tag{3}$$

The study of $s(t)$ has been hampered by its relative numerical intractability. Only when $a(t)$ is a pure exponential is a closed form answer available. We take $a(t)$ to be

$$a(t) = 2(e^{-t} - e^{-2t}) \tag{4}$$

i.e., a(t) is the density of the sum of two independent exponentially distributed variables, one with parameter 1, the other with parameter 2. The Laguerre or Erlang coefficients for a(t) are written down immediately using (2a) or (2b). By inspection, we see that the best value of v for the Erlang coefficients is 1.5, and that the best value of v for the Laguerre coefficients will be between 1 and 2 (turned out to be about 1.3).

To perform multiple convolution, we make use of (1.13) and (4.6), which state that for either Laguerre or Erlang building blocks we have $T^{\#}_{f*g} = T^{\#}_f T^{\#}_g$. Hence $T^{\#}_h = (T^{\#}_a)^m$, where $h(t) = a^{(m)}(t)$ and we have the following algorithm for the multiple convolution of a(t) using Laguerre building blocks:

I. Generate or store in the computer the Laguerre coefficients $\{a^+_n\}^N_0$ for a(t). Note that this is a finite set.

II. Convert $\{a^+_n\}$ to $\{a^{\#}_n\}$. Since $T^{\#}_a(u) = (1-u)T^+_a$, this corresponds to a simple differencing operation.

III. Perform m-fold discrete convolution on $\{a^{\#}_n\}^N_0$. Retain only the first N+1 terms in each convolution. The result is the sequence $\{h^{\#}_n\}^N_0$.

IV. Convert $\{h^{\#}_n\}$ to $\{h^+_n\}$. This is the inverse of the differencing operation, i.e., summation.

V. Sum the series $\sum_0^N h^+_n \ell_n(t)$ to get h(t).

A completely analogous procedure is used for the Erlang building blocks. A few remarks may be helpful:

(1) The steps involve a finite number of computer operations on discrete sequences. The operations are things that a computer does well. The steps are easy to program and are typically done with high speed and accuracy.

(2) In applications, step III was done using the formula

$$(f*g)_n = \sum_0^n f_k g_{n-k} \quad .$$

Since N is typically less than 100 and frequently less than 50, the use of sophisticated techniques, e.g., Fast Fourier Transform, for the discrete convolution does not seem warranted.

(3) A typical sequence of operations is to make a trial run with N = 30, repeat for various values of v to get the most rapid convergence, then increase N as needed to get the desired accuracy at all necessary values of t. Thus, working with the computer in an interactive mode is ideal for this methodology.

With this algorithm, the functions $a^{(n+1)}(t)$ may be computed for desired values of t, starting with $a^{(1)}(t) = a(t)$. Figure 1 displays s(t) for $\lambda = .2$ and $\lambda = 1$. For both the Laguerre and Erlang building blocks, accuracy was best near the origin and decreased with increasing values of t. Three digit accuracy, i.e., sufficient to plot figure 1, was gotten by using N = 30 for all the convolutions (either set of building blocks) and using the first 6 terms of the infinite sum for s(t). When high accuracy was desired, the Erlang

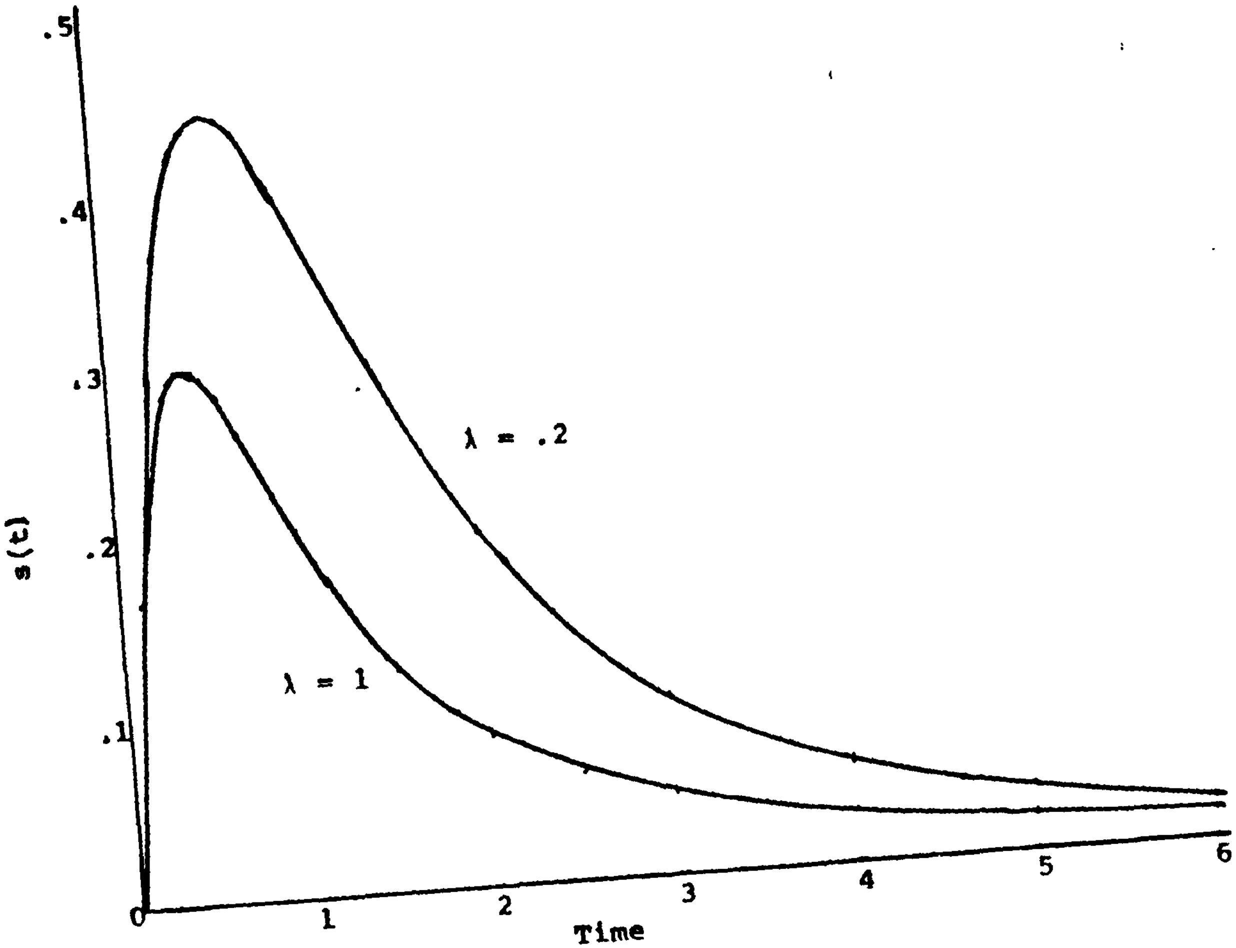

Fig. 1: s(t) versus t

building blocks proved superior to the Laguerre building blocks. Using N = 40 for all convolutions, accuracy to about 10 digits after the decimal point was attained out to $t = 10$ using 23 terms in the infinite sum expression for $s(t)$.

The service time distribution has mean 2/3, so that for $\lambda = .2$, $s(t)$ represents an honest density, i.e., it integrates to unity. For $\lambda > 2/3$, the formula for $s(t)$ remains valid but integrates to less than unity since there is positive probability that the queue will never empty after the first customer arrives.

Example 2: Solving a renewal equation

This example illustrates the transform methodology in the solution of a Volterra integral equation of type 2, e.g., a renewal equation. The use of series methods in the solution of renewal equations has been used with some success by previous authors, e.g., [8], [16], who have typically restricted themselves to the Taylor series. As this paper makes clear, the use of the Taylor series is a special case of a much more general approach.

In a reliability context, suppose an item has failure time density $a(t)$, and suppose that the maintenance policy is to replace the item at random times, whether it has failed or not. Let the replacement times be exponentially distributed with rate parameter λ, and let $s(t)$ be the density of the resulting effective failure time distribution. Then probabilistic reasoning gives

$$s(t) = b(t)*\{\delta(t) + c(t) + c^{(2)}(t) + \dots\}$$

$$= b(t)*\sum_{0}^{\infty} c^{(k)}(t) \tag{5}$$

where $$b(t) = a(t)\exp(-\lambda t) \tag{6a}$$

$$c(t) = \lambda\exp(-\lambda t) \int_{t}^{\infty} a(y)dy \tag{6b}$$

and $\delta(t)$ is the Dirac delta function. It is clear that $s(t)$ obeys a renewal equation

$$s(t) = b(t) + c(t)*s(t) \tag{7}$$

The multiple convolutions in (5) could be evaluated as in the preceding example, but the direct solution of the renewal equation (7) is much more economical and accurate.

To solve (7), we note that (7), (1.12), and (4.5b) imply that

$$T_s^{\#} = T_b^{\#} + T_c^{\#}T_s^{\#} \quad ,$$

which implies the discrete renewal equation

$$\{s_n^{\#}\} = \{b_n^{\#}\} + \{c_n^{\#}\}*\{s_n^{\#}\} \tag{8}$$

i.e., $$s_n^{\#} = b_n^{\#} + \sum_{0}^{n} c_k^{\#}s_{n-k}^{\#} \quad . \tag{9}$$

Equation (9) can be solved for $s_n^{\#}$ recursively, starting with $s_0^{\#} = b_0^{\#}$, providing that $c_0^{\#} \neq 1$. The algorithm for solving (7) is hence steps

I - V of Example 1, except that step III requires the solution of the discrete renewal equation (8).

We choose $a(t) = 2(e^{-t} - e^{-2t})$ as in Example 1 so that (6a), (6b) become

$$b(t) = 2[e^{-(1+\lambda)t} - e^{-(2+\lambda)t}] \tag{10a}$$

$$c(t) = 2\lambda e^{-(1+\lambda)t} - \lambda e^{-(2+\lambda)t} \quad . \tag{10b}$$

The Laguerre or Erlang coefficients for b(t) and c(t) may be written down immediately by using (2a), (2b), respectively. The resulting s(t) is displayed in figure 2 for $\lambda = .2$ and $\lambda = 4$. As expected, the mean time to complete the message transmission increases as the mean interruption rate increases. Using N = 40, both sets of building blocks give about 4 digits of accuracy after the decimal point in the time interval [0, 10] for $\lambda = 4$. For $\lambda = .2$ and N = 40, we get 5 and 10 digit accuracy, respectively, for the Laguerre and Erlang building blocks.

Example 3: The deconvolution problem

The deconvolution problem involves solving an equation of form

$$b(t) = a(t)*f(t) \tag{11}$$

with a(t), b(t) known and f(t) unknown. This problem is discussed in section 8, and equation (8.4) provides an algorithm for solving the discrete deconvolution equation (8.2), which is of the form

$$\{b_n^\#\} = \{a_n^\#\}*\{f_n^\#\} \quad . \tag{12}$$

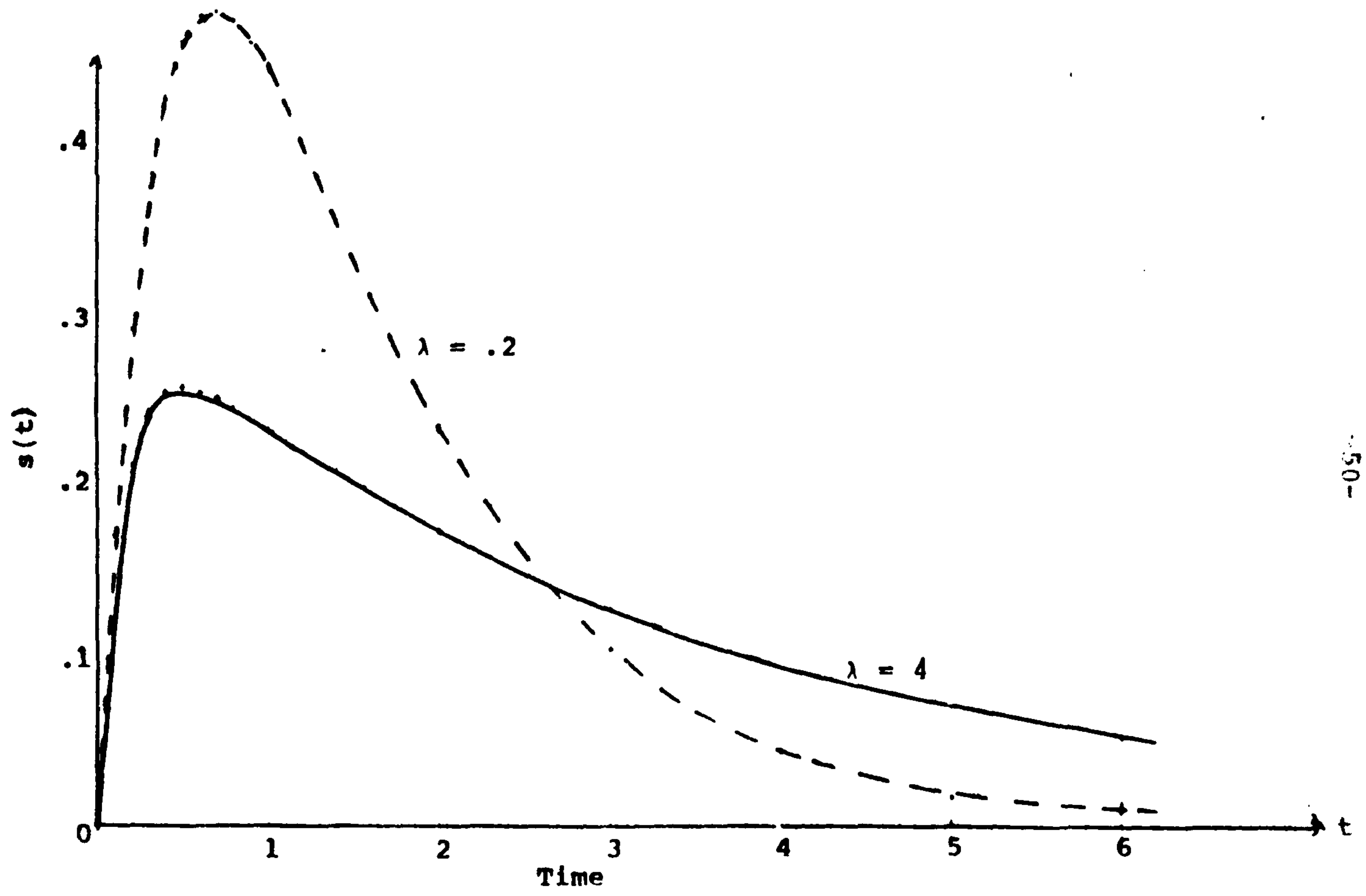

Fig. 2: s(t) versus t

It should be noted that (12) can be recast as a renewal equation, i.e.,

$$\{f_n^{\#}\} = \{b_n^{\#}\} + [\delta(n) - \{a_n^{\#}\}]*\{f_n^{\#}\}$$

where $\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$.

Hence, a subroutine for solving a discrete renewal equation also serves to solve the discrete deconvolution equation.

In an M/M/1 queue setting, let $s_n(t)$ be the density of the time for the queue to empty, given a start with n customers. It is known [9] that $s_n(t)$ satisfies

$$g_{-n}(t) = s_n(t)*g_0(t) \tag{13}$$

where $$g_n(t) = (\lambda/\mu)^{\frac{1}{2}n}e^{-(\lambda+\mu)t}I_n(2\sqrt{\lambda\mu}t),\ n = 0,\pm 1,\ldots \tag{14}$$

with λ the arrival rate and μ the service rate. The density $s_n(t)$ is known [9] to be equal to $(n/t)g_{-n}(t)$, but we solve (13) numerically to illustrate the deconvolution technique.

When $\lambda = \mu = \frac{1}{2}$, (14) simplifies to

$$g_n(t) = e^{-t}I_n(t) \quad .$$

We expand $I_n(t)$ in a Taylor's series and immediately get a natural expression of $g_n(t)$ as an Erlang series with $v = 1$. The deconvolution

algorithm proceeds with no difficulty and figure 3 displays $s_1(t)$. The density is integrable, but has a very heavy tail, i.e., it has no moment. Twenty terms in the series provides more than enough accuracy to make the plot, but more terms are needed for large values of t. Sixty terms gave about 5 digits of accuracy after the decimal point at $t = 50$, with 10 digits at $t = 20$.

When $\lambda = .25$ and $\mu = 1$, (14) becomes

$$g_n(t) = (\tfrac{1}{2})^n e^{-1.25t} I_n(t) \quad .$$

Again, the expansion of $I_n(t)$ in a Taylor's series provides a natural expression of $g_n(t)$ as an Erlang series, this time with $v = 1.25$. Figure 3 also displays $s_1(t)$ for this case. The tail is very light and 30 terms in the series gives an accuracy of about 8 digits out to $t = 10$.

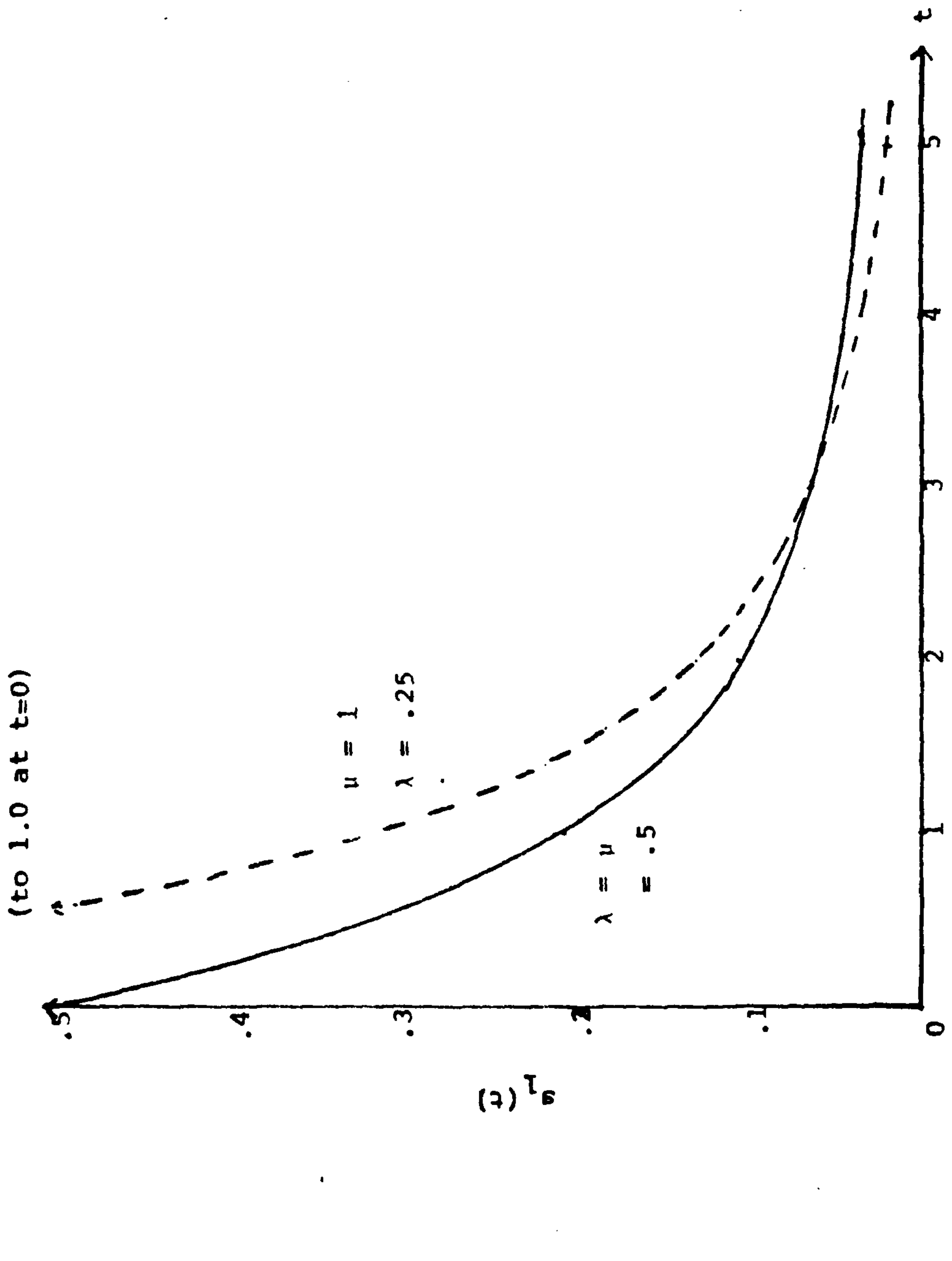

Fig. 3: $s_1(t)$ versus time

10. Interpolation Methods and Problems

For the solution of integral equations of form (0.1) or (0.2) where the given functions a(x) and b(x) are only known numerically at some finite set of points, interpolation procedures are needed. Interpolation of the data via spline functions may or may not be suitable for the methods we have described. The discontinuities in the derivatives at the knots intrinsic to those methods preclude the Erlang transform methods of Section 4 since a(x) and b(x) cannot then be integral functions. Laguerre transforms are possible, but the functions will not be rapidly decreasing and hence, as shown in Section 6, the Laguerre transform coefficients are not rapidly decreasing and the tails of $a_n^\dagger$ could be excessively long. A sufficiency of data points could permit spline interpolation of high order, and corresponding accuracy. The difficulties surrounding such finite data set problems, and the accuracy of our methods when applied to such problems, require further study.

11. Generalization of the Method to Other Families of Function

The basic method can be extended to other classes of functions. Consider any sequence of functions $(a_n(\tau))_o^\infty$ with Laplace transforms $(\alpha_n(s))_o^\infty$ for which $\alpha_n(s)$ has the form

$$\alpha_n(s) = \alpha_o(s) \quad (\beta(s))^n \ , \tag{1}$$

Suppose further that under the change of variable $w = \beta(s)$, $\alpha_o(s)$ becomes $p(w)$, i.e., $\alpha_o(\beta^{-1}(w)) = p(w)$, where $p(w)$ is regular at $w = 0$. Let $g(t) = \sum g_n a_n(t)$, $h(t) = \sum h_n a_n(t)$. Then $L\left[g(t)\right] = \sum g_n \alpha_n(s) = \alpha_o(s) \ \sum g_n \left[\beta_o(s)\right]^n = p(w) \ \sum g_n w^n$, and $L\left[h(t)\right] = p(w) \sum h_n w^n$. In our previous notation, we then have for $T_g^\dagger(w) = \sum g_n w^n$

$$p(w) \ T^\dagger_{g*h} = p(w) \ T_g^\dagger(w) \ p(w) \ T_h^\dagger(w). \tag{2}$$

Hence

$$T^\#_{g*h}(w) = T^\#_g(w) \ T^\#_h(w) \tag{3}$$

where $T_g^\#(w) = p(w) \ T_g^\dagger(w)$; and the framework for computation developed in Section 1 for the Laguerre transform carries over.

There are many classes of functions $\{a_n(t)\}$ having the necessary algebraic structure for $\alpha_n(s)$. Two such classes quite different in form from the Erlang class and Laguerre class are of interest. In the first the building blocks are the functions

$$a_n(t) = \frac{n}{\sqrt{4\pi t^3}} \exp\{-n^2/4t\} \tag{4}$$

which arise in the theory of diffusion. For these functions one has

$$\alpha_n(s) = e^{-n\sqrt{s}} \tag{5a}$$

$$\beta(s) = e^{-\sqrt{s}}, \quad \alpha_o(s) = p(w) = 1. \tag{5b}$$

The second family has Bessel function building blocks

$$a_n(t) = J_n(t), \tag{6}$$

for which [1]

$$\alpha_n(s) = \{\sqrt{s^2+1} - s\}^n/\sqrt{s^2+1} \tag{7a}$$

$$\beta(s) = \{\sqrt{s^2+1} - s\} \tag{7b}$$

$$\alpha_o(s) = (s^2+1)^{-\frac{1}{2}}; \; p(w) = 2w/(1+w^2). \tag{7c}$$

For any of these families $(a_n(t))$ we may use instead $(a_n(t)e^{-\gamma\tau})$, since the necessary properties are preserved.

In special contexts, such as underwater sound signals, a particular family might be more appropriate than the Erlang family or Laguerre family. Each family of functions has special advantages and special limitations and computational algorithms appropriate to that family are needed.

12. Inversion of Laplace Transforms

The operational framework we have developed for convolutions also provides formulae for inverting Laplace transforms in terms of the building blocks. Two such formulae of general interest are presented in this section. Other formulae may be obtained from Section 11 for special classes of functions.

When $\alpha(s) = \int_0^\infty e^{-st} a(t)d\tau$ is regular at infinity and vanishes there, inversion of $\alpha(s)$ as a series is implicit in Theorem 4.1. This may be worth stating formally. The result is given in a more restrictive setting by D. V. Widder [20].

Theorem 4.3

If: (a) $\alpha(s)$ is regular at ∞ and vanishes there

(b) $\alpha(s-\frac{1}{2}) = \sum_0^\infty c_m (\frac{1}{s})^{m+1}$

then $a(\tau)$ is entire and given by

$$a(\tau) = \sum_0^\infty c_m \frac{\tau^m}{m!} \quad e^{-\tau/2}$$

where the series is absolutely convergent for all τ.

Note that the Taylor expansion of $T_a^{L+}(u)$ about $u = 0$ together with (1.4) also provides an explicit inversion formula for Laplace transforms. Specifically, one has:

Theorem 4.4

Let $a(\tau) \in L_2(0,\infty)$ and let $\alpha(s) = L[a(\tau)]$. Then $a(\tau)$ is given by

$$(13) \qquad a(\tau) = \sum_{m=0}^{\infty} \left\{ \frac{1}{m!} \left(\frac{d}{du}\right)^m \left[\frac{1}{1-u}\, \alpha\left(\frac{1}{2}\, \frac{1+u}{1-u}\right)\right]\Big|_{u=0} \ell_m(\tau) \right\} .$$

Numerical inversion of Laplace transforms via numerical integration techniques has received considerable attention, e.g., in the book by Bellman, Kalaba, and Lockett [2].† Such techniques do not appear to exploit analytically founded accuracy as fully as our procedures and are placed in jeopardy by ill-conditioning [2, p. 33]. An extensive comparison of the two procedures is needed.

The idea of inverting Laplace transforms via the operational tools of Laguerre functions goes back, to the authors' knowledge, to Lanczos in his book, Applied Analysis [11]. The presentation there is somewhat obscure and the scope of the procedure and its legitimacy are essentially undeveloped. Feller presents explicit inversion formulae for Laplace transforms [6] of largely theoretical value. Neither Feller nor Widder nor Bellman refer to Lanczos' method.

Acknowledgment. The authors wish to thank H. Ross for helpful comments, U. Sumita for his extensive and generous technical assistance, and Ms. L. Ziegenfuss for editorial contributions.

†The reader may wish to examine the papers by A. Papoulis, "A new method of inversion of the Laplace transform," Q. Ap. Math. 14, 1957 and R. V. Churchill, "The inversion of the Laplace transformation by a direct expansion in series," Math. Zeitschrift 42, 1937.

Appendix A

Classical properties of the Laguerre polynomials $L_n(x)$ and functions $\ell_n(x) = L_n(x)e^{-x/2}$ (cf. [1])

1. $\int_0^\infty e^{-x}L_n(x)L_m(x)dx = \delta_{mn}$

2. $\int_0^\infty \ell_n(x)\ell_m(x)dx = \delta_{mn}$

3. $\sum_0^\infty u^n L_n(x) = (1-u)^{-1}\exp\{-\frac{xu}{1-u}\}$

4. $\sum_0^\infty u^n \ell_n(x) = (1-u)^{-1}\exp\{-\frac{1}{2}\, x\, \frac{1+u}{1-u}\}$

5. $L_n(x) = \sum_{k=0}^{n} (-1)^k \frac{1}{k!} \binom{n}{k} x^k$

6. $x^n = \sum_{k=0}^{n} (-1)^k \frac{1}{k!} \binom{n}{k} L_k(x)$

7. $\int_0^\infty e^{-sx}\ell_n(x)dx = \frac{1}{s+\frac{1}{2}} \left(\frac{s-\frac{1}{2}}{s+\frac{1}{2}}\right)^n \; ; \quad \int_0^\infty \ell_n(x)dx = (-1)^n\cdot 2$

8. $L_n(0) = \ell_n(0) = 1$

9. $\ell_n(x) \sim \frac{x^n}{n!} e^{-x/2}$, $x \to \infty$

10. $L_n(x) = \frac{1}{n!} e^x \left(\frac{d}{dx}\right)^n \{x^n e^{-x}\}$

11. $\sum_0^\infty \ell_n(x) \frac{u^n}{n!} = e^u e^{-\frac{1}{2}x} J_0(2\sqrt{xu})$

12. $x\ell_n(x) = (2n+1)\ell_n(x) - (n+1)\ell_{n+1}(x) - n\ell_{n-1}(x)$

13. $-\ell_n'(x) = \frac{1}{2}\ell_n(x) + \sum_{0}^{n-1} \ell_m(x)$

APPENDIX B

A TABLE OF LAGUERRE TRANSFORMS

$a(\tau)$	$\alpha(s)$	$T_a^{\#}(u)$	$T_a^{\dagger}(u)$	$a_n^{\#}$	$a_n^{\dagger}$
$a(\tau)$	$\int_0^\infty e^{-s\tau} a(\tau)\, d\tau$	$\sum_{n=0}^{\infty} a_n^{\#} u^n = \alpha\left(\frac{1+u}{2(1-u)}\right)$	$\sum_{n=0}^{\infty} a_n^{\dagger} u^n = (1-u)^{-1} T_a^{\#}(u)$	$a_n^{\dagger} - a_{n-1}^{\dagger}$	$\int_0^\infty a(\tau)\, l_n(\tau)\, d\tau$
$e^{-\theta\tau}$ $(\theta > 0)$	$(s+\theta)^{-1}$	$\frac{2(1-u)}{(1+2\theta)+(1-2\theta)u}$	$\frac{2}{(1+2\theta)+(1-2\theta)u}$	$e_{\theta n} = -\frac{4}{(2\theta+1)^2}\left(\frac{2\theta-1}{2\theta+1}\right)^{n-1}$	$\frac{2}{(2\theta+1)}\left(\frac{2\theta-1}{2\theta+1}\right)^{n}$
$\exp\left[-\frac{(1+\lambda)}{2(1-\lambda)}\tau\right]$ $(\lvert\lambda\rvert < 1)$	$\left[s + \frac{(1+\lambda)}{2(1-\lambda)}\right]^{-1}$	$\frac{(1-\lambda)(1-u)}{(1-\lambda u)}$	$\frac{(1-\lambda)}{(1-\lambda u)}$	$-(1-\lambda)^2 \lambda^{n-1}$	$(1-\lambda)\lambda^n$
$e^{-\omega t}$ $(\omega = A+iB \quad A>0)$	$(s+\omega)^{-1}$	$\frac{2(1-u)}{(1+2\omega)+(1-2\omega)u}$	$\frac{2}{(1+2\omega)+(1-2\omega)u}$	$-\frac{4}{(2\omega+1)^2}\left(\frac{2\omega-1}{2\omega+1}\right)^{n-1}$	$\frac{2}{(2\omega+1)}\left(\frac{2\omega-1}{2\omega+1}\right)^{n}$
$Re(e^{-\omega\tau}) = e^{-A\tau}\cos B\tau$					$Re\left[\frac{2}{(2\omega+1)}\left(\frac{2\omega-1}{2\omega+1}\right)^{n}\right]$
$Im(e^{-\omega\tau}) = -e^{-A\tau}\sin B\tau$					$Im\left[\frac{2}{(2\omega+1)}\left(\frac{2\omega-1}{2\omega+1}\right)^{n}\right]$
$l_k(\tau)$	$\frac{1}{(s+\frac{1}{2})}\left(\frac{s-\frac{1}{2}}{s+\frac{1}{2}}\right)^k$	$(1-u)u^k$	u^k	$\delta_{nk} - \delta_{n-1,k}$	$\delta_{n,k}$
$\frac{e^{-\theta\tau}\tau^M}{M!}$ $(\theta>0)$	$(s+\theta)^{-M-1}$	$\left[\frac{2(1-u)}{(1+2\theta)+(1-2\theta)u}\right]^{M+1}$	$\frac{1}{(1-u)} T_n^{\#}(u)$	$\left\{e_{\theta r}\right\}_n^{(M+1)}$	$\sum_{m=0}^{n} a_m^{\#}$
$e_M(\tau) = \frac{e^{-\frac{\tau}{2}}\tau^M}{M!}$	$(s+\frac{1}{2})^{-M-1}$	$(1-u)^{M+1}$	$(1-u)^M$	$\binom{M+1}{n}(-1)^n;\ 0 \le n \le M+1$ $0;\ n > M+1$	$\binom{M}{n}(-1)^n;\ 0 \le n \le M$ $0;\ n > M$
$e^{-\frac{\tau}{2}} J_0(2\sqrt{\lambda\tau})$ $(\lambda>0)$	$\frac{1}{(s+\frac{1}{2})}\exp\left[-\frac{\lambda}{s+\frac{1}{2}}\right]$	$(1-u)e^{-\lambda(1-u)}$	$e^{-\lambda(1-u)}$	$\frac{e^{-\lambda}\lambda^{n-1}(\lambda-n)}{n!}$	$\frac{e^{-\lambda}\lambda^n}{n!}$
$e^{-\frac{3}{2}\tau} I_0(\tau)$	$\left[(s+\frac{3}{2})(s+\frac{1}{2})\right]^{-\frac{1}{2}}$	$(1-u)(3-2u)^{-\frac{1}{2}}$	$(3-2u)^{-\frac{1}{2}}$	$-\frac{(n+1)}{3\sqrt{3}\, n\, 6^{n-1}}\binom{2n-2}{n-1}$	$\frac{1}{\sqrt{3}\, 6^n}\binom{2n}{n}$

APPENDIX C

OPERATIONAL PROPERTIES OF LAGUERRE TRANSFORMS

$a(\tau)$	$\alpha(s)$	$T_a^*(u)$	$T_a^\dagger(u)$	a_n^*	$a_n^\dagger$
$a(\tau)$	$\int_0^\infty e^{-s\tau} a(\tau)\,d\tau$	$\sum_{n=0}^\infty a_n^* u^n = \alpha\left(\frac{1+u}{2(1-u)}\right)$	$\sum_{n=0}^\infty a_n^\dagger u^n = (1-u)^{-1} T_a^*(u)$	$a_n^\dagger - a_{n-1}^\dagger$	$\int_0^\infty a(\tau)\, \ell_n(\tau)\, d\tau$
$f(\tau)$	$\varphi(s)$	$T_f^*(u) = \sum_{n=0}^\infty f_n^* u^n$	$T_f^\dagger(u) = \sum_{n=0}^\infty f_n^\dagger u^n$	$f_n^\dagger - f_{n-1}^\dagger$	$f_n^\dagger = \int_0^\infty f(\tau)\, \ell_n(\tau)\, d\tau$
$g(\tau)$	$\gamma(s)$	$T_g^*(u) = \sum_{n=0}^\infty g_n^* u^n$	$T_g^\dagger(u) = \sum_{n=0}^\infty g_n^\dagger u^n$	$g_n^\dagger - g_{n-1}^\dagger$	$g_n^\dagger = \int_0^\infty g(\tau)\, \ell_n(\tau)\, d\tau$
$f(\tau) * g(\tau)$	$\varphi(s)\cdot\gamma(s)$	$T_f^*(u)\; T_g^*(u)$	$(1-u)^{-1}\, T_f^*(u)\, T_g^*(u)$	$\sum_{m=0}^{n} f_{n-m}^*\, g_m^*$	$\sum_{k=0}^{n} \sum_{m=0}^{k} f_{k-m}^*\, g_m^*$
$\frac{d}{d\tau} f(\tau)$	$s\varphi(s) - f(0)$	$\frac{(1+u)}{2(1-u)} T_f^*(u) - f(0)$	$\frac{(1+u)}{2(1-u)^2} T_f^*(u) - \frac{f(0)}{(1-u)}$	$\frac{1}{2}\left(f_n^\dagger + f_{n-1}^\dagger\right) - f(0)\,\delta_{n0}$	$\frac{1}{2} f_n^\dagger - \sum_{m=n}^\infty f_m^\dagger$
$\tau f(\tau)$	$-\frac{d}{ds}\varphi(s)$			$(3n+1) f_n^\dagger - (n+1) f_{n+1}^\dagger - (3n-1) f_{n-1}^\dagger + (n-1) f_{n-2}^\dagger$	$(2n+1) f_n^\dagger - (n+1) f_{n+1}^\dagger - n f_{n-1}^\dagger$
$\int_\tau^\infty f(y)\,dy$	$\frac{1}{s}\left[\varphi(0) - \varphi(s)\right]$	$\frac{2(1-u)}{(1+u)}\left[T_f^*(-1) - T_f^*(u)\right]$	$\frac{2}{(1+u)}\left[T_f^*(-1) - T_f^*(u)\right]$	$-2 f_n^* + 4 \sum_{m=0}^\infty (-1)^m f_{n+m}^*$	$-2 \sum_{m=0}^\infty (-1)^m f_{n+1+m}^*$

EXPLANATION OF SOME ENTRIES IN THE TABLE OF OPERATIONAL PROPERTIES OF LAGUERRE TRANSFORMS

(Note)

For notational convenience, any element with a negative subscript is considered to be zero in the following discussion.

[1]

$$\left.\begin{array}{l} a(\tau) = \frac{d}{d\tau} f(\tau) \\ \sum_{n=0}^{\infty} |f_n^\dagger| < \infty \end{array}\right\rangle \Rightarrow \left\langle \begin{array}{l} a_n^\dagger = \frac{1}{2} f_n^\dagger - \sum_{m=n}^{\infty} f_m^\dagger \\ a_n^* = \frac{1}{2}(f_n^\dagger + f_{n-1}^\dagger) - f(0)\,\delta_{n,0} \end{array}\right.$$

(proof)

$$a(\tau) = \frac{d}{d\tau} f(\tau) = \sum_{n=0}^{\infty} f_n^\dagger \frac{d}{d\tau} \ell_n(\tau) = -\sum_{n=0}^{\infty} f_n^\dagger \left[\frac{1}{2}\ell_n(\tau) + \sum_{r=0}^{n-1} \ell_r(\tau)\right] = \sum_{n=0}^{\infty}\left[\frac{1}{2} f_n^\dagger - \sum_{m=n}^{\infty} f_m^\dagger\right] \ell_n(\tau)$$

$$a_0^* = a_0^\dagger = \frac{1}{2} f_0^\dagger - \sum_{m=0}^{\infty} f_m^\dagger = \frac{1}{2} f_0^\dagger - f(0)$$

$$a_n^* = a_n^\dagger - a_{n-1}^\dagger = \frac{1}{2} f_n^\dagger - \sum_{m=n}^{\infty} f_m^\dagger - \frac{1}{2} f_{n-1}^\dagger + \sum_{m=n-1}^{\infty} f_m^\dagger = \frac{1}{2}(f_n^\dagger + f_{n-1}^\dagger) \quad \text{for } n \geq 1$$

[2]

$$\left.\begin{array}{l} a(\tau) = \tau f(\tau) \\ f(\tau) \in L_2[0,\infty) \end{array}\right\rangle \Rightarrow \left\langle \begin{array}{l} a_n^\dagger = (2n+1) f_n^\dagger - (n+1) f_{n+1}^\dagger - n f_{n-1}^\dagger \\ a_n^* = (3n+1) f_n^\dagger - (n+1) f_{n+1}^\dagger - (3n-1) f_{n-1}^\dagger + (n-1) f_{n-2}^\dagger \end{array}\right.$$

(proof)

$$a(\tau) = \tau f(\tau) = \sum_{n=0}^{\infty} f_n^\dagger \tau \ell_n(\tau) = \sum_{n=0}^{\infty} f_n^\dagger \left[(2n+1)\ell_n(\tau) - (n+1)\ell_{n+1}(\tau) - n\ell_{n-1}(\tau)\right]$$

$$= \sum_{n=0}^{\infty}\left[(2n+1) f_n^\dagger - (n+1) f_{n+1}^\dagger - n f_{n-1}^\dagger\right] \ell_n(\tau)$$

$$a_0^* = a_0^\dagger = f_0^\dagger - f_1^\dagger$$

$$a_n^* = a_n^\dagger - a_{n-1}^\dagger = (2n+1) f_n^\dagger - (n+1) f_{n+1}^\dagger - n f_{n-1}^\dagger - (2n-1) f_{n-1}^\dagger + n f_n^\dagger + (n-1) f_{n-2}^\dagger$$

$$= (3n+1) f_n^\dagger - (n+1) f_{n+1}^\dagger - (3n-1) f_{n-1}^\dagger + (n-1) f_{n-2}^\dagger$$

[3]

$$\left.\begin{array}{l} a(\tau) = \int_\tau^\infty f(y)\,dy \\ \sum_{n=0}^{\infty} |f_n^\dagger| < \infty \end{array}\right\rangle \Rightarrow \left\langle \begin{array}{l} a_n^\dagger = -2 \sum_{m=0}^{\infty} (-1)^m f_{n+1+m}^* \\ a_n^* = -2 f_n^* + 4 \sum_{m=0}^{\infty} (-1)^m f_{n+m}^* \end{array}\right.$$

(proof)

$$a(\tau) = \int_0^\infty f(y)\,dy - \int_0^\tau f(y)\,dy \quad \therefore \alpha(p) = \frac{1}{p}\left[\varphi(0) - \varphi(p)\right] \quad \text{where } \varphi(p) = \int_0^\infty e^{-p\tau} f(\tau)\,d\tau$$

$$T_a^*(u) = \alpha\left(\frac{1+u}{2(1-u)}\right) = \frac{2(1-u)}{(1+u)}\left[\varphi(0) - \varphi\left(\frac{1+u}{2(1-u)}\right)\right] = \frac{2(1-u)}{(1+u)}\left[T_f^*(-1) - T_f^*(u)\right]$$

$$T_a^\dagger(u) = \frac{2}{1+u}\left[T_f^*(-1) - T_f^*(u)\right] = -\frac{2}{1+u}\sum_{n=0}^{\infty} f_n^*\left[u^n - (-1)^n\right] = -2\sum_{n=0}^{\infty} f_n^* \sum_{m=0}^{n-1} (-1)^{n-1-m} u^m$$

$$\therefore a_n^\dagger = -2 \sum_{r=n+1}^{\infty} (-1)^{r-1-n} f_r^* = -2 \sum_{m=0}^{\infty} (-1)^m f_{n+1+m}^*$$

$$a_n^* = a_n^\dagger - a_{n-1}^\dagger = -2 \sum_{m=0}^{\infty} (-1)^m \left[f_{n+1+m}^* - f_{n+m}^*\right] = -2\left[f_n^* - 2\sum_{m=0}^{\infty} (-1)^m f_{n+m}^*\right]$$

$$= -2 f_n^* + 4 \sum_{m=0}^{\infty} (-1)^m f_{n+m}^*$$

References

[1] Abramowitz, M. and Stegun, I. A. (1965). Handbook of Mathematical Functions, Dover, NY.

[2] Bellman, R., Kalaba, R. E. and Lockett, J. A. (1966). Numerical Inversion of the Laplace Transform, American Elsevier, NY.

[3] Cleroux, R. and McConalogue, D. J. (1976). "A Numerical Algorithm for Recursively Defined Convolution Integrals Involving Distribution Functions," Management Science, 22, 1138-1146.

[4] Cox, D. R. (1955). "A Use of Complex Probabilities in the Theory of Stochastic Processes," Proc. Camb. Phil. Soc., 51, 313-319.

[5] Dym, H and McKean, H. P. (1972). Fourier Series and Integrals, Academic Press, NY.

[6] Feller, W. (1966). An Introduction to Probability Theory and its Applications, Vol. II, Wiley, New York.

[7] Guilleman, E. (1963). Theory of Linear Physical Systems, John Wiley.

[8] Hammersly, J. M. (1958). pp. 287-88 in discussion of paper by W. L. Smith, "Renewal Theory and its Ramifications," J. Roy. Stat. Soc. (B), 20, 243-302.

[9] Keilson, J. (1965). Green's Function Methods in Probability Theory, Charles Griffin, London.

[10] Keilson, J., Kester, A. and Waterhouse, C., "A Circulatory Model for Human Metabolism," J. Theoretical Biology (to appear).

[11] Lanczos, C. (1964). Applied Analysis, Prentice-Hall.

[12] McCully, J. (1956). The Operational Calculus of the Laguerre Transform, Thesis, University of Michigan, University Microfilms International, Ann Arbor, Michigan.

[13] Neuts, M. F. (1977). "Algorithms for the Waiting Time Distribution," Algorithmic Methods in Probability, North-Holland, New York.

[14] Neuts, M. F. (1975). "Computational Uses of the Method of Phases in the Theory of Queues," Computers and Mathematics with Applications, 1, 151-166.

[15] Riordan, J. (1968). Combinatorial Identities, Wiley, New York.

[16] Smith, W. L. and Leadbetter, M. R. (1963). "On the Renewal Function for the Weibull Distribution," Technometrics, 5, 393-396.

[17] Szego, G. (1975). Orthogonal Polynomials, Amer. Math. Soc., Providence, RI.

[18] Takacs, L. (1962). Introduction tc the Theory of Queues, Oxford.

[19] Tricomi, F. G. (1957). Integral Equations, Wiley.

[20] Widder, D. V. (1971). An Introduction to Transform Theory, Academic Press, NY.

PART 2

THE BILATERAL LAGUERRE TRANSFORM

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0. Introduction and Summary

In a previous paper, hereafter designated by [A], a description was given of a Laguerre transformation which maps a function $f(\tau)$ in $L_2(0,\infty)$ into a sequence $(f_n^{\#})_0^{\infty}$ on the nonnegative integers. Moreover, for two such functions $f(\tau)$, $g(\tau)$, the convolution $f(\tau)*g(\tau)$ is mapped into the lattice convolution $(f*g)_n^{\#} = \sum_0^n f_{n-m}^{\#} g_m^{\#} = ((f_m^{\#})*(g_m^{\#}))_n$. One obtains thereby an algorithmic basis for the computation of multiple convolutions $f^{(k)}(\tau)$ and related infinite series of importance to statistics and applied probability.

Such Laguerre transforms have one-sided functions as their natural domain because the Laguerre polynomials $L_n(\tau)$ and Laguerre functions $\ell_n(\tau) = L_n(\tau)e^{-\tau/2}$ are associated with the one-sided weight function $e^{-\tau}$ on $(0,\infty)$. Nevertheless, the methods have a simple extension to two-sided functions on the full continuum $(-\infty, \infty)$ via the same Laguerre functions as we will see.

A variety of applications exist to statistics, operations research, and engineering. In statistics, for example, one has need for multiple convolutions of two-sided distributions unavailable analytically, that of the logistic distribution, for example. Even relatively innocuous distributions such as the Laplace distribution convolve with difficulty.

In operations research studies dealing with queues, inventories and storage systems, one encounters as a structural entity [3] the extended renewal density $h(x) = \sum_1^{\infty} a^{(k)}(x)$, where $a(x)$ is a probability density function with two-sided support. For many densities of interest, evaluation of $h(x)$ has been resistant.

In the earlier paper [A], the crucial role of the complex plane in the formulation of the algorithms was evident, even though the algorithms were entirely in the real domain. For the bilateral transform, the complex plane is again very much present, with Laurent expansions, bilateral Laplace transformation and conformal mapping entering as crucial tools.

The first section extends the earlier formalism to the full continuum. That this extension is natural, and not just an artificial piecing together of the formalism for each half-line, will be clear from (1.9), (1.12) and (1.13). The harmony of the basis will also emerge vividly in Section 3, which deals with the extent of the transform coefficients, and associated uncertainty relations. The topic of extent is crucial to the utility of the Laguerre transform method as a numerical tool. Numerical examples are presented in Section 5. A table of contents provides the reader with an overview of the paper.†

†Two references (V. I. Krylov and N. S. Skoblya [8], and W. T. Weeks [12]) have come to the authors' attention subsequent to publication of [A]. Both deal with the use of Laguerre functions for the numerical inversion of one-sided Laplace transforms.

1. The bilateral Laguerre transform

In this section the one-sided methodology of [A] will be recast in a natural way to provide a corresponding representation of two-sided functions. In this extended setting the one-sided functions will be a special subcase.

Let $f(\tau)$ be any function in $L_2(-\infty,\infty)$. Let $U(\tau) = 1$, $\tau \geq 0$, $U(\tau) = 0$, $\tau < 0$. Then $f(\tau) = f_+(\tau) + f_-(\tau)$ where $f_+(\tau) = f(\tau)U(\tau)$ is in $L_2(0,\infty)$ and $f_-(\tau) = f(\tau)U(-\tau)$ is in $L_2(-\infty, 0)$. The discrepancy at $\tau = 0$ may be ignored. Clearly, one may write, employing the notation in [A],

$$(1.1) \qquad f_+(\tau) = \sum_0^\infty f^\dagger_{n+}\ell_n(\tau)U(\tau)$$

$$(1.2) \qquad f_-(\tau) = \sum_0^\infty f^\dagger_{n-}\ell_n(-\tau)U(-\tau)$$

and

$$(1.3) \qquad f(\tau) = \sum_{-\infty}^\infty f^\dagger_m h_m(\tau) \quad , \quad \tau \neq 0 \ ,$$

where

$$(1.4a) \qquad h_m(\tau) = \ell_m(\tau)U(\tau) \ , \qquad m \geq 0$$

$$(1.4b) \qquad h_m(\tau) = -\ell_{-m-1}(-\tau)U(-\tau) \ , \qquad m < 0 \ .$$

The puzzling minus sign in (1.4b) plays an important role which will emerge soon. We see that

$$(1.4c) \qquad h_m(\tau) = -h_{-m-1}(-\tau) \ , \quad \text{all } m,\tau \ .$$

Note also from (1.1), (1.2) and (1.3) that

$$(1.4d) \qquad f_n^\dagger = f_{n+}^\dagger \ , \quad n \geq 0 \ ; \quad f_n^\dagger = -f_{(-n-1)-}^\dagger \ , \quad n < 0 \ .$$

The set of functions $\{h_m(\tau)\}_{-\infty}^{\infty}$ form a complete orthonormal system for $L_2(-\infty,\infty)$. Consequently, one has

$$(1.5) \qquad f_m^\dagger = \int_{-\infty}^{\infty} f(\tau)h_m(\tau)d\tau = \begin{cases} \int_0^\infty f(\tau)\ell_m(\tau)d\tau \ , & m \geq 0 \\ -\int_0^\infty f(-\tau)\ell_{-m-1}(\tau)d\tau \ , & m < 0 \end{cases} \ .$$

From the orthonormality, one has the Parseval relation

$$(1.6) \qquad \int_{-\infty}^{\infty} f^2(\tau)d\tau = \sum_{-\infty}^{\infty} f_m^{\dagger 2} \ .$$

We know that Bilateral-Laplace transformation gives for $n \geq 0$ (cf. [A]),

$$(1.7) \qquad L_B[\ell_n(\tau)U(\tau)] = \int_{-\infty}^{\infty} e^{-s\tau} \ell_n(\tau)U(\tau)d\tau$$

$$= (s - 1/2)^n/(s + 1/2)^{n+1} \ , \quad \mathrm{Re}(s) > -1/2$$

and

$$(1.8) \qquad L_B[-\ell_n(-\tau)U(-\tau)] = (s + 1/2)^n/(s - 1/2)^{n+1} \ , \quad \mathrm{Re}(s) < 1/2 \ .$$

It then follows from (1.4a) and (1.4b) that, <u>for all n</u> (explaining the minus sign remark below (1.4b)),

$$(1.9a) \qquad L_B[h_n(\tau)] = (s - 1/2)^n/(s + 1/2)^{n+1} \ ;$$

$$-\infty < n < \infty \ ; \quad -1/2 < \mathrm{Re}(s) < 1/2 \ .$$

From (1.4) and (1.9) we have the interesting identity

$$(1.9b) \qquad h_m(\tau)*h_n(\tau) = h_0(\tau)*h_{m+n}(\tau) \; ; \quad \text{all } m,n,\tau$$

which may be compared with eq. (1.9) of [A]. For $f(\tau) \in L_2(-\infty,\infty)$, with $(f_n^\dagger) \in \ell_1$, i.e., $\sum_{-\infty}^{\infty} |f_n^\dagger| < \infty$, we have formally from (1.3) and (1.9a)

$$(1.10) \qquad \phi_B(s) \overset{\text{def}}{=} L_B[f(\tau)] = \sum_{-\infty}^{\infty} f_n^\dagger \left(\frac{s - 1/2}{s + 1/2}\right)^n \frac{1}{s + 1/2} \quad .$$

A discussion of the domain of validity of (1.10) will be given in the next section. The ideas and results are a two-sided extension of the corresponding one-sided results presented in [A]. As we will see, the series in (1.10) will be absolutely convergent in some strip $-A_- < \mathrm{Re}(s) < A_+$, $A_- > 0$, $A_+ > 0$, when $f_+(\tau)$ and $f_-(\tau)$ are sufficiently smooth and "rapidly decreasing" as in [A]. It then follows from (1.10), that in the corresponding domain D in the complex u-plane, containing $\{u: |u| = 1\}$, one will have for the dagger generating function $T_f^\dagger(u)$

$$(1.11) \qquad T_f^\dagger(u) \overset{\text{def}}{=} \sum_{-\infty}^{\infty} f_m^\dagger u^m = \frac{1}{1-u}\, \phi_B\left(\frac{1}{2}\, \frac{1+u}{1-u}\right) \quad ,$$

and

$$(1.12) \qquad T_f^\#(u) \overset{\text{def}}{=} (1-u) T_f^\dagger(u) = \phi_B\left(\frac{1}{2}\, \frac{1+u}{1-u}\right) \quad .$$

For bilateral convolution it is known that $L_B[f(\tau)*g(\tau)] = \phi_B(s)\gamma_B(s)$. Hence we have from (1.12),

$$(1.13) \qquad T_{f*g}^\#(u) = T_f^\#(u) T_g^\#(u) \quad .$$

It then follows that as in [A], bilateral convolution may be mapped into lattice convolution via the two-sided Laguerre transform

$$f_n^{\#} = \int_{-\infty}^{\infty} f(\tau)[h_n(\tau) - h_{n-1}(\tau)]d\tau ,$$

i.e.,

(1.14) $$f_n^{\#} = f_n^{\dagger} - f_{n-1}^{\dagger} .$$

The validity of (1.10) in a convergence strip containing the imaginary axis in its interior will permit $T_f^{\#}(u)$ to be regular in an annulus containing the unit circle in its interior, when $\phi_B(s)$ is regular at infinity as we will see in Section 2. The situation is somewhat modified in the absence of such regularity.

Note that $f(\tau)$ even, i.e., $f(\tau) = f(-\tau)$ implies that $\phi_B(s)$ is even in s. It then follows from (1.12) that

(1.15a) $$f(\tau) = f(-\tau) \Longleftrightarrow f_n^{\#} = f_{-n}^{\#} ,$$

i.e., $f(\tau)$ even in τ implies that $f_n^{\#}$ is even in n. Similarly, one has

(1.15b) $$f(\tau) = -f(-\tau) \Longleftrightarrow f_n^{\#} = -f_{-n}^{\#} .$$

The reader will verify from (1.1) and (1.2) that when $f(\tau)$ is even (so that $f_+(\tau) = f_-(-\tau)$), then $f_{n+}^{\dagger} = f_{n-}^{\dagger}$. It then follows from (1.4d) that

(1.16a) $$f(\tau) = f(-\tau) \Longleftrightarrow f_n^{\dagger} = -f_{(-n-1)}^{\dagger}$$

(1.16b) $$f(\tau) = -f(-\tau) \Longleftrightarrow f_n^{\dagger} = f_{(-n-1)}^{\dagger} ,$$

i.e., the daggers have their symmetry about n = - 1/2.

It is clear from the simplicity of the form of (1.13) and from the relative simplicity of the symmetry relations (1.15a,b) for $f_n^{\#}$ over those of (1.16a,b) for $f_n^{\dagger}$ that the sharp coefficients $f_n^{\#}$ are a more natural vehicle for our algorithms than the daggar coefficients $f_n^{\dagger}$. Nevertheless, the latter are of algorithmic and theoretical importance. Indeed, they are needed for a final inversion $f(\tau) = \sum_{-\infty}^{\infty} f_n^{\dagger} h_n(\tau)$ returning to the function sought.

As in the introductory paper [A], the general setting for the transform is the $L_2(-\infty,\infty)$ class for which $\sum_{-\infty}^{\infty} f_n^{\dagger 2} < \infty$, so that $(f_n)_{-\infty}^{\infty} \in \ell_2$. As for the one-sided functions, however, the methodology is of value algorithmically only for functions sufficiently smooth, e.g., "rapidly decreasing" (cf. §2). Summability of $(f_n^{\dagger})$ is of special importance. We note, therefore, that

$$(1.17) \qquad (f_n^{\dagger})_{-\infty}^{\infty} \in \ell_1 \Rightarrow (f_n^{\#})_{-\infty}^{\infty} \in \ell_1 \Rightarrow \sum_{-\infty}^{\infty} f_n^{\#} = 0$$

as seen immediately from (1.14), and $\sum_{A}^{B} f_n^{\#} = f_B^{\dagger} - f_{A-1}^{\dagger}$. Moreover,

$$(1.18) \qquad (f_n^{\dagger})_{-\infty}^{\infty} \in \ell_1 \Rightarrow f_n^{\dagger} = \sum_{-\infty}^{n} f_m^{\#} = -\sum_{n+1}^{\infty} f_m^{\#} \quad ,$$

and (1.18) then permits one to go from $(f_n^{\#})$ to $(f_n^{\dagger})$. We also note from (1.4d) and (1.14) that for all $(f_n^{\dagger}) \in \ell_2$

$$(1.19) \qquad f_n^{\#} = \begin{cases} f_{n+}^{\#} \, , & n > 0 \quad . \\ f_{0+}^{\#} + f_{0-}^{\#} \, , & n = 0 \quad . \\ f_{(-n)-}^{\#} \, , & n < 0 \quad . \end{cases}$$

Consequently one has

$$T_f^{\#}(u) = \sum_{-\infty}^{\infty} f_n^{\#} u^n = \sum_{-\infty}^{0} f_{(-n)-}^{\#}\, u^n + \sum_{0}^{\infty} f_{n+}^{\#} u^n \ ,$$

i.e.,

$$T_f^{\#}(u) = T_{f+}^{\#}(u) + T_{f-}^{\#}(u^{-1}) \quad . \tag{1.20}$$

§2. Rate of convergence of the Laguerre coefficients; regularity structure in the complex s and u planes

A. *Structure for the one-sided case*

The bilateral Laguerre transform will be a useful tool for the mechanization of convolution if the sequences $(f_n^{\dagger})_{-\infty}^{\infty}$ and $(f_n^{\#})_{-\infty}^{\infty}$ fall off rapidly as in [A]. The definitions employed there will be repeated for convenience.

DEF 2.0a. $(f_n)_0^{\infty} \in C_{\downarrow}^{\infty}(N+) \overset{\text{def}}{\Longleftrightarrow} n^K|f_n| \to 0,\ n \to \infty$ for all non-negative integers K.

DEF 2.0b. $f(\tau) \in C_{\downarrow}^{\infty}(R+) \overset{\text{def}}{\Longleftrightarrow} |\tau^q (\frac{d}{d\tau})^p f(\tau)| < M_{q,p}$ for all non-negative integers q,p.

One then says $(f_n)_0^{\infty}$ *is a rapidly decreasing sequence and* $f(\tau)$ *is a* rapidly decreasing function on $R+$.

If, in particular, $f_+(\tau)$ is "rapidly decreasing", then as in [A], Theorem 6.6, $(f_n^{\dagger})_0^{\infty}$ and $(f_n^{\#})_0^{\infty}$ will be rapidly decreasing, e.g., one will have $n^K|f_n^{\dagger}| \to 0$, $n \to \infty$ all positive integers. K. Similarly, if $f_-(-\tau)$ is rapidly decreasing on $(0,\infty)$, then $(f_{-n-1}^{\dagger})_0^{\infty}$ and $(f_{-n-1}^{\#})_0^{\infty}$ will also be rapidly decreasing.

A systematic development of the extension of these ideas to the two-sided setting will be given soon.

P2.1

$$(f_n^{\dagger})_0^{\infty} \in C_{\downarrow}^{\infty}(N+) \overset{A}{\Longleftrightarrow} (f_n^{\#})_0^{\infty} \in C_{\downarrow}^{\infty}(N+) \overset{B}{\Longleftrightarrow} f(\tau) \in C_{\downarrow}^{\infty}(R+)$$

<u>Proof:</u> $\overset{A}{\Rightarrow}$ is immediate from $f_n^{\#} = f_n^{\dagger} - f_{n-1}^{\dagger}$ in (1.14).

$$\overset{A}{\Leftarrow}: \quad n^K|f_n^{\dagger}| = n^K\left|-\sum_{n+1}^{\infty} f_m^{\#}\right| \le \frac{1}{n}\sum_{n+1}^{\infty} m^{K+1}|f_m^{\#}| < \frac{C_K}{n} \to 0 \text{ as } n \to \infty$$

for all K .

$\overset{B}{\Leftrightarrow}$: Restatement of Theorem 6.6 in [A]. □

We need to relate our smoothness conditions to a radius of convergence for $T_f^{\#}(u)$. We define the following set.

<u>DEF. 2.2</u>

$$G_R = \{(f_n)_0^{\infty} : \sum_{n=0}^{\infty} |f_n|R^n < \infty\} \quad .$$

We then have

<u>P2.3a.</u> If a sequence has a radius of convergence larger than unity, then the sequence is rapidly decreasing, i.e.,

$$G_R \subset C_{\downarrow}^{\infty}(N_+) \text{ , any } R > 1 \quad . \tag{2.1}$$

<u>Proof</u>

$$n^K|f_n^{\dagger}| = \left(\frac{n^K}{R^n}\right)|f_n^{\dagger}|R^n < C\,\frac{n^K}{R^n} \to 0 \text{ as } n \to \infty \text{ . } \square$$

<u>P2.3b.</u> $(f_n^{\dagger})_0^{\infty} \in G_R \Longleftrightarrow (f_n^{\#})_0^{\infty} \in G_R$, for any $R > 1$.

<u>Proof:</u> The direction => follows at once from (1.14). The direction <= is an immediate consequence of $T_f^{\dagger}(u) = (1-u)^{-1}T_f^{\#}(u)$ relating the generating functions. Since $(f_n^{\#})_0^{\infty} \in G_R$, $R > 1 \Rightarrow T_f^{\#}(u)$ is regular for $0 < |u| < R$, the point $u = 1$ lies in the domain of regularity of $T_f^{\#}(u)$. Moreover, $T_f^{\#}(1) = 0$. It then follows that $u = 1$ is a regular point for $T_f^{\dagger}(u)$, etc. □

Remark 2.3c. We note that $C_{\downarrow}^{\infty}(N+) \subset G_1$, and is a proper subset of G_1, as may be seen from $(f_n)_0^{\infty}$ with $f_n = \dfrac{1}{1+n^2}$. We also note that the union of the sets G_R with $R > 1$ is a proper subset of $C_{\downarrow}^{\infty}(N+)$, as in the example $f_n = e^{-\sqrt{n}}$.

We will see that over-convergence of $T_f^{\#}(u) = \sum_0^{\infty} f_n^{\#} u^n$, i.e., the availability of a radius of convergence greater than one is associated with over convergence of the Laplace transforms $\phi(s)$ in the presence of a simple condition at infinity.

P2.4. Let $f(\tau) \in L_2(0,\infty)$ with $\phi(s)$, $T_f^{\#}(u)$ defined for $\text{Re}(s) > 0$, $|u| < 1$. Then for the analytic continuation one has

$$\{\phi(s) = \int_0^{\infty} e^{-s\tau} f(\tau)d\tau \text{ is regular at } s = \infty \text{ and } \phi(\infty) = 0\} \Longleftrightarrow$$

$$\{T_f^{\#}(u) \text{ is regular at } u = 1 \text{ and } T_f^{\#}(1) = 0\} \quad .$$

Proof: From (1.12), $T_f^{\#}(u) = \phi(\frac{1}{2}\,\frac{1+u}{1-u})$ and $\phi(s) = T_f^{\#}(\frac{2s-1}{2s+1})$. By setting $w = \frac{1}{s}$, $\phi(\frac{1}{w}) = T_f^{\#}(\frac{2-w}{2+w})$. Thus $\phi(\frac{1}{w})$ is regular at $w = 0$ and vanishes there if and only if $T_f^{\#}(u)$ is regular at $u = 1$ and $T_f^{\#}(1) = 0$. □

The situation is significantly modified when $\phi(s)$ is not regular at infinity. Some simple bilinear mappings from the s-plane to the u-plane are needed to understand the modified behavior. We state the basic results which follow from standard results of conformal mapping theory [9].

P2.5.

(A) Let $s(u) = \frac{1}{2}\,\frac{1+u}{1-u}$ and $A = \{u: R_1 \le |u| \le R_2\}$, where $0 \le R_1 \le 1 \le R_2$. Then A maps into s(A) in the complex s-plane, the complement of the two disjoint circles possibly tangent, as shown in Fig. 2.1.

(B) Let $u(s) = \frac{2s - 1}{2s + 1}$ and $B = \{s: R_1 \le |s| \le R_2\}$, where $0 \le R_1 \le \frac{1}{2} \le R_2$. Then B maps into u(B) in the complex u-plane, as shown in Fig. 2.2.

(C) Let $u(s) = \frac{2s - 1}{2s + 1}$ and $D = \{s: \delta_1 \le Re(s) \le \delta_2\}$, where $\delta_1 \le 0 \le \delta_2$. Then u(D) for two key cases is as shown in Fig. 2.3a,b.

Two key theorems for the one-sided functions may now be stated.

Theorem 2.6

Let $T_f^{\#}(u) = \phi(\frac{1}{2}\,\frac{1+u}{1-u})$. Then the following (A) and (B) are equivalent.

(A) $(f_n^{\#})_0^{\infty} \in G_R$ for some $R > 1$.

(B) $\phi(s)$ is regular in $\{s: \delta < Re(s)\}$ for some $\delta < 0$. $\phi(s)$ is also regular at $s = \infty$ and $\phi(\infty) = 0$.

Proof

The theorem follows from P2.5 and simple argument in the complex plane. Details are omitted.□

If the regularity condition of $\phi(s)$ at $s = \infty$ is dropped in (B) of Theorem 2.6, $T_f^{\#}(u)$ is no longer guaranteed to be regular at $u = 1$. The resulting regularity structure is described next. The proof is as before with details omitted.

Theorem 2.7

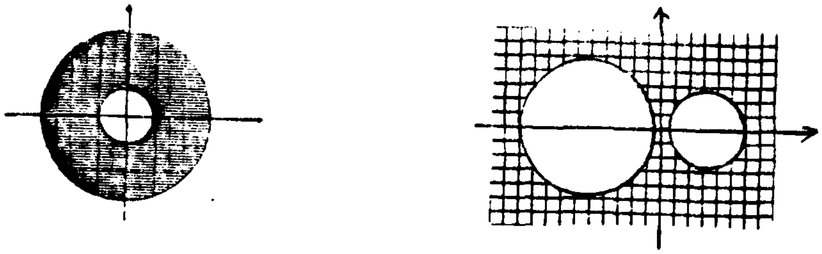
Let $T_f^{\#}(u) = \phi(\frac{1}{2}\,\frac{1+u}{1-u})$. Then

(1) The following (A) and (B) are equivalent.

(A) $T_f^{\#}(u)$ is regular inside a circle c^* containing every point of the set $\{u: |u| = 1, u \ne 1\}$ in its interior, with the circle c^* tangent to the unit circle at $u = 1$.

(B) $\phi(s)$ is regular in $E_\delta = \{\delta: \delta < Re(s)\}$ for some $\delta < 0$.

Fig. 2.1

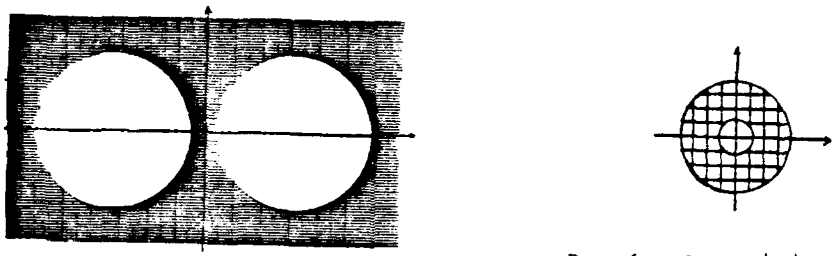


$A = \{u: R_1 \le |u| \le R_2\}$

$0 \le R_1 \le 1 \le R_2$

$s(A)$ where $s(u) = \frac{1}{2}\frac{1+u}{1-u}$

Fig. 2.2



$u(B)$ where $u(s) = \frac{2s-1}{2s+1}$

$B = \{s: R_1 < |s| < R_2\}$

$0 \le R_1 \le \frac{1}{2} \le R_2$

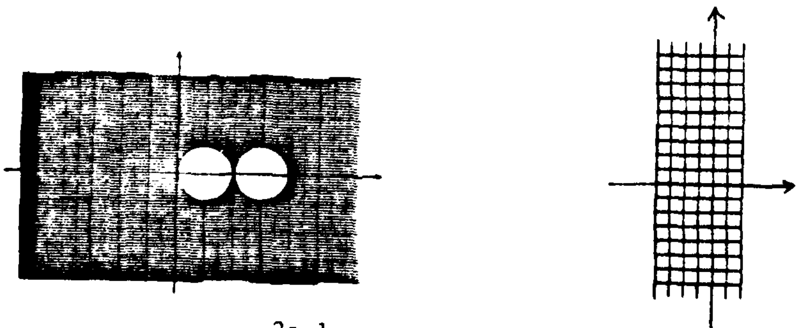
Fig. 2.3a

$u(D)$ where $u(s) = \frac{2s-1}{2s+1}$

$D = \{s: \delta_1 < \text{Re}(s) < \delta_2\}$

$-\frac{1}{2} < \delta_1 < 0 < \delta_2$

Fig 2.3b



$u(D)$ where $u(s) = \frac{2s-1}{2s+1}$

$D = \{s: \delta_1 < \text{Re}(s) < \delta_2\}$

$\delta_1 < -\frac{1}{2} < 0 < \delta_2$

Complex u-plane

Complex s-plane

(2) Let $s \in E_\delta$ be such that $s = S + iT$. Then (B) implies $\lim_{T\to\pm\infty} \phi(S + iT) = \lim_{u\to 1} T_f^{\#}(u) = 0$, the limit being taken along the unit circle.

Corollary 2.8

If $\phi(s)$ is entire, then $T_f^{\#}(u)$ has only one singularity at $u = 1$. This corollary also can be seen directly from $T_f^{\#}(u) = \phi(\frac{1}{2}\frac{1+u}{1-u})$.

B. Structure for the two-sided case

Theorems 2.6 and 2.7 can be easily extended to the two-sided case in a natural way. First we give the following definition.

DEF

$$\text{(2.2a)} \qquad f(\tau) \in C_{\downarrow}^{*}(R_1) \overset{\text{def}}{\iff} f_+(\tau) \in C_{\downarrow}^{\infty}(R_+),\ f_-(-\tau) \in C_{\downarrow}^{\infty}(R_+)$$

$$\text{(2.2b)} \qquad (f_n)_{-\infty}^{\infty} \in C_{\downarrow}^{*}(N) \overset{\text{def}}{\iff} (f_n)_0^{\infty} \in C_{\downarrow}^{\infty}(N+),\ (f_{-n})_0^{\infty} \in C_{\downarrow}^{\infty}(N+) .$$

Remark 2.9: The definition implicit in (2.2a) requires no continuity at $\tau = 0$ for $f(\tau)$. The asterisk distinguishes this class from the ordinary class $C_{\downarrow}^{\infty}(R_1)$ of rapidly decreasing functions which are differentiable to all order at every τ.

These definitions with P2.1 yield the following immediately.

P2.10

$$\text{(2.3)} \qquad (f_n^{\dagger})_{-\infty}^{\infty} \in C_{\downarrow}^{*}(N) \iff (f_n^{\#})_{-\infty}^{\infty} \in C_{\downarrow}^{*}(N) \iff f(\tau) \in C_{\downarrow}^{*}(R) .$$

We also have as a simple extension of the one-sided case:

P2.11

$$(f_n^{\dagger})_{-\infty}^{\infty} \in \ell_1 \Rightarrow (f_n^{\#})_{-\infty}^{\infty} \in \ell_1 \text{ and } \sum_{-\infty}^{\infty} f_n^{\#} = 0 .$$

Proof

$(f_n^{\dagger})_{-\infty}^{\infty} \in \ell_1$ implies that $(f_{n+}^{\dagger})_0^{\infty} \in \ell_1$ and $(f_{n-}^{\dagger})_0^{\infty} \in \ell_1$ and that

$(f_n^{\#})_{-\infty}^{\infty} \in \ell_1$.

Since $T_f^{\#}(u) = T_{f+}^{\#}(u) + T_{f-}^{\#}(u^{-1})$ from (1.20),

(2.4) $$\sum_{n=-\infty}^{\infty} f_n^{\#} = \sum_{n=0}^{\infty} f_{n+}^{\#} + \sum_{n=0}^{\infty} f_{n-}^{\#} = 0 \quad . \quad \square$$

The extension of Theorems 2.6 and 2.7 to the two-sided case now follows.

Theorem 2.12

Let $T_f^{\#}(u) = \phi_B(\frac{1}{2}\,\frac{1+u}{1-u})$. Then the following (A) and (B) are equivalent.

(A) $T_f^{\#}(u)$ is regular in $A_R = \{u\colon R_- < |u| < R_+\}$ where $0 < R_- < 1 < R_+$.

(B) $\phi_B(s)$ is regular in $E = \{s\colon -A_- < \mathrm{Re}(s) < A_+\}$ where $A_- > 0$, $A_+ > 0$, as well as at $s = \infty$, where $\phi_B(\infty) = 0$.

Proof

Details omitted.$\square$

We note that from P2.3a,b and P2.10 that either (A) or (B) of Theorem 2.12 implies $f(\tau) \in C_+^*(R)$. The two-sided counterpart of Theorem 2.7 is given next.

Theorem 2.13

Let $T_f^{\#}(u) = \phi_B(\frac{1}{2}\,\frac{1+u}{1-u})$. Then

(1) the following (A) and (B) are equivalent.

(A) $T_f^{\#}(u)$ is regular inside the region $C^{**} = C_2^* - C_1^*$ where:

(a) C_2^* is a circle containing every point of the set

$\{u: |u| = 1, u \neq 1\}$ in its interior;

(b) C_1^* is a circle where $C_1^* - \{1\}$ is contained in the interior of the set $\{u: |u| \leq 1\}$;

(c) both C_1^* and C^* are tangent to the unit circle at $u = 1$.

(B) $\phi_B(s)$ is regular in $E = \{s: -A_- < \mathrm{Re}(s) < A_+\}$ where $A_- > 0$ and $A_+ > 0$.

(2) Let $s = S + iT$ for $s \in E$. Then (B) implies that $\lim_{T \to \pm\infty} \phi_B(S + iT) = = \lim_{u \to 1} T_f^{\#}(u) = 0$, the limit being taken along the unit circle.

Proof:

Omitted.□

C. Rapidly decreasing functions on the full continuum

P2.10 says that $f(\tau) \in C_{\downarrow}^*(R)$ if and only if $(n^K f_n^{\dagger}) \in \ell_1$ for all nonnegative integer K, or equivalently $(n^K f_n^{\#}) \in \ell_1$ for all nonegative integer K. But this permits the discontinuity of $(\frac{d}{dt})^r f(\tau)$ at $\tau = 0$. When $(\frac{d}{dt})^r f(\tau)$ is continuous at $\tau = 0$ for all $r = 0,1,2,\ldots$, one finds surprisingly that all the moments of $f_n^{\dagger}$ and $f_n^{\#}$ vanish. We remind the reader that $\ell_n(t)$ satisfies the operator relation (cf. Eq. (6.2) of [A])

$$\text{(2.5a)} \qquad L[\ell_n(\tau)] = (n + \tfrac{1}{2})\ell_n(\tau)$$

where

$$\text{(2.5b)} \qquad L = [\tfrac{1}{4}\tau - \frac{d}{d\tau}\tau\frac{d}{d\tau}] \quad .$$

From (1.4a) and (2.5) we have $L[h_n(\tau)] = (n + \frac{1}{2})h_n(\tau)$ for $n \geq 0$. For $n < 0$, $L[h_n(\tau)] = -L[\ell_{-n-1}(-\tau)U(-\tau)] = -(-n - 1 + \frac{1}{2})\ell_{-n-1}(-\tau)U(-\tau) = (n + \frac{1}{2})h_n(\tau)$. Consequently one has

$$(2.6)\qquad L^r[h_n(\tau)] = (n+\tfrac{1}{2})^r h_n(\tau) \text{ for all } n \text{ and } r = 0,1,2,\ldots \ .$$

We also verify trivially from equation (1.4a,b) that

$$(2.7)\qquad \theta_n = h_n(0+) - h_n(0-) = 1 \quad \text{for all } n \ .$$

It follows that for $f(\tau) \in C_\downarrow^\infty(R_1)$,

$$(2.8)\qquad L^r f(\tau)\Big|_{0-}^{0+} = \sum_{n=-\infty}^{\infty} (n+\tfrac{1}{2})^r f_n^\dagger \ , \ r = 0,1,2,\ldots \ ,$$

where $a(\tau)\Big|_{0-}^{0+} = a(0+) - a(0-)$. Legitimacy is assured by $(n^k f_n^\dagger)_{-\infty}^{\infty} \in \ell_1$ for all $k \geq 0$ (cf. Theorem 6.6 of [A]), the uniform boundedness of $\ell_n^{(k)}(\tau)$ on R_1 (cf. Lemma 6.2 of [A]) and the dominated convergence theorem. Similarly, when $(f_n^\dagger)_{-\infty}^{\infty} \in C_\downarrow^*(N)$, (2.8) will be valid. The following theorem can now be proven.

<u>Theorem 2.14</u>

Let

$$(2.9)\qquad A = \{f(\tau) : (f_n^\dagger)_{-\infty}^{\infty} \in C_\downarrow^*(N) \ , \quad \sum_{-\infty}^{\infty} n^k f_n^\dagger = 0 \ , \quad k = 0,1,2,\ldots\} \ .$$

Then

$$C_\downarrow^\infty(R_1) = A \ .$$

We note that A is a proper subset of $C_\downarrow^*(R_1)$ which is equal to $\{f(\tau) : (f_n^\dagger)_{-\infty}^{\infty} \in C_\downarrow^*(N)\}$. It differs in that its elements have all moments equal to 0. The class $C_\downarrow^\infty(R_1)$ contains real analytic functions of great interest in mathematical statistics and probability theory, e.g., $e^{-x^2/2}$ and $(\cosh x)^{-1}$. The vanishing of $\sum_{-\infty}^{\infty} n^k f_n^\dagger$ and as we will see of $\sum_{-\infty}^{\infty} n^k f_n^\#$ for such functions is of corresponding interest. Two lemmas will be employed to prove the theorem.

Lemma 2.15

$$\sum_{-\infty}^{\infty} n^k f_n^{\dagger} = 0,\ k = 0,1,2,\ldots \iff \sum_{-\infty}^{\infty} n^k f_n^{\#} = 0,\ k = 0,1,2,\ldots$$

Proof

(=>) Immediate from $f_n^{\#} = f_n^{\dagger} - f_{n-1}^{\dagger}$ and the binomial theorem.

(<=) One has

$$\frac{T_f^{\dagger}(u)}{(1-u)^k} = \frac{T_f^{\#}(u)}{(1-u)^{k+1}} \quad \text{for} \quad u = e^{i\theta}\ ,\ \theta \neq 0\ . \tag{2.10}$$

The limit of the expression on the right exists and equals 0 as $u \to 1$ by repeated application of L'Hospital's rule. A similar application of L'Hospital's rule to the expression on the left, then gives via induction on K, $\sum_{-\infty}^{\infty} n^K f_n^{\dagger} = 0$ for all K as required.□

Lemma 2.16

$$f(\tau) \in A \Rightarrow \begin{cases} \text{(a)} & \dfrac{d}{d\tau} f(\tau) \text{ exists at } \tau = 0 \\ \text{(b)} & \dfrac{d}{d\tau} f(\tau) \in A \\ \text{(c)} & (\dfrac{d}{d\tau})^r f(\tau) \in A \end{cases}$$

Proof

We first show that $f(\tau) \in A$ implies that $\frac{d}{d\tau} f(\tau)$ exists at $\tau = 0$. This is easy since from (2.5b) and (2.8)

$$-\frac{d}{d\tau} f(\tau)\Big|_{0-}^{0+} = L[f(\tau)]\Big|_{0-}^{0+} = \sum_{-\infty}^{\infty} (n + \tfrac{1}{2}) f_n^{\dagger} = 0\ .$$

Since $f(\tau) \in C_+^*(R_1)$ implies that $f(\tau)$ is differentiable in $(-\infty, 0)$ and $(0, \infty)$, differentiability on R_1 follows.

We must now show (b) of the Lemma, i.e., that $\frac{d}{d\tau} f(\tau) \in A$. One has

for continuously differentiable $f(\tau)$ that

$$(2.11) \qquad [\frac{d}{d\tau} f(\tau)]_n^{\#} = \frac{1}{2}(f_n^{\dagger} + f_{n-1}^{\dagger}) \quad ,$$

from Theorem 4.4 of Section 4. From (2.11) and the binomial theorem, we have

$$(2.12) \qquad \sum_{-\infty}^{\infty} n^K [\frac{d}{d\tau} f(\tau)]_n^{\#} = \frac{1}{2} \sum_{-\infty}^{\infty} n^K f_n^{\dagger} + \frac{1}{2} \sum_{-\infty}^{\infty} \{(n-1) + 1\}^K f_{n-1}^{\dagger} = 0 \quad .$$

Finally, from (2.12) and Lemma 2.15, one has

$$\sum_{-\infty}^{\infty} n^K [\frac{d}{d\tau} f(\tau)]_n^{\dagger} = 0 \ , \quad K = 0,1,2,\ldots \quad \text{proving (b).}$$

Since (c) follows immediately by induction, the Lemma is proven.□

Proof of Theorem 2.14

($\Rightarrow$) $f(\tau) \in C_{\downarrow}^{\infty}(R_1) \Rightarrow L^r f(\tau) \in C_{\downarrow}^{\infty}(R_1)$ so that $L^r f(\tau)\Big|_{0-}^{0+} = 0$. Thus $\sum_{-\infty}^{\infty} (n + \frac{1}{2})^r f_n^{\dagger} = 0$ from (2.8), whence $\sum_{-\infty}^{\infty} n^K f_n^{\dagger} = 0$ for all K.

($\Leftarrow$) It is easy to see that

$$(2.13) \qquad f(\tau) \in C_{\downarrow}^{\infty}(R_1) \iff \begin{cases} f(\tau) \in C_{\downarrow}^{*}(R_1) \\ (\frac{d}{d\tau})^r f(\tau)\Big|_{0-}^{0+} = 0, \ r = 0,1,2,\ldots \end{cases} \quad .$$

Consequently we need only verify that every derivative of $f(\tau)$ is continuous at $\tau = 0$. The theorem therefore follows from Lemma 2.16 and (2.13).□

Remark 2.17: It is natural to ask how, in the setting of Theorem 2.13, the continuity of $f(\tau)$ and all its derivatives at $\tau = 0$ modifies the behavior of $\phi_B(s)$ in its convergence strip. The answer is given by Theorem (a) of §2.2 in Dym and McKean [1] which says that the Fourier transform of a function in $C_{\downarrow}^{\infty}(R_1)$ is also in $C_{\downarrow}^{\infty}(R_1)$. For us this means that $\phi^*(u) = \phi_B(iu)$ is in

$C_{\downarrow}^{\infty}(R_1)$, i.e., that $u^p(\frac{d}{du})^q\phi^*(u) \to 0$ as $|u| \to \infty$. It is clear from $\int_{-\infty}^{\infty} e^{-s\tau}e^{-\alpha\tau}f(\tau)d\tau = \phi_B(s+\alpha)$ that every function $\phi_B(\alpha + iu)$ for α in the interval of absolute convergence of the bilateral Laplace transform will also be rapidly decreasing.

Remark 2.18: A discussion of rate of convergence of the Laguerre transform coefficients for functions having only finite number of drivatives may be developed along the line of Section 6 of [A]. This development will be omitted.

3. The extent of the Laguerre transform coefficients

The accuracy of the algorithm based on the Laguerre transform depends on one's ability to represent the functions present compactly, i.e., with vectors $(f_n^\dagger)_{-K_-}^{K_+}$ of reasonable length. Correspondingly, some measure of extent of the sequence $(f_n^\dagger)_{-K_-}^{K_+}$ may be useful and feeling for the relationships between the extent of the approximating sequence and that of the function helpful. In general, there are inverse relationships between these two extents as in the Heisenberg inequality of Fourier transforms (cf. Dym and McKean [1], §2.8). In what follows, we discuss only rapidly decreasing functions $f(\tau)$ in either $C_\downarrow^*(R)$ or $C_\downarrow^\infty(R)$. This condition may be weakened, as mentioned in Remark 2.18.

We first exhibit such inequality for the dagger transform $(f_n^\dagger)_{-\infty}^{\infty}$. To do so, we require the following notation. Let $\tilde{N}_1^\dagger$ be the extent of the dagger transform $(f_n^\dagger)_{-\infty}^{\infty}$ defined by

$$(3.1) \qquad \tilde{N}_1^\dagger = \sum_{-\infty}^{\infty} |n + 1/2| f_n^{\dagger 2} / \sum_{-\infty}^{\infty} f_n^{\dagger 2} \quad .$$

Similarly, let $\tilde{T}_1$ be the extent of the original function $f(\tau)$, i.e.,

$$(3.2) \qquad \tilde{T}_1 = \int_{-\infty}^{\infty} |\tau| f^2(\tau) d\tau / \int_{-\infty}^{\infty} f^2(\tau) d\tau \quad .$$

Finally, let

$$(3.3) \qquad \tilde{B}_1 = \int_{-\infty}^{\infty} |\tau| f'(\tau)^2 d\tau / \int_{-\infty}^{\infty} f^2(\tau) d\tau \quad .$$

Then one has the following theorem.

Theorem 3.1

Let $f(\tau) \epsilon C_{\downarrow}^{*}(R)$. Then

(1) $\bar{N}_1^{\dagger} = \frac{1}{4}\bar{T}_1 + \bar{B}_1$;

(2) $\bar{N}_1^{\dagger} \geq \frac{1}{4}(\bar{T}_1 + \frac{1}{\bar{T}_1}) \geq \frac{1}{2}$.

Proof

From (2.6), $L[f(\tau)] = \sum\limits_{-\infty}^{\infty} (n + \frac{1}{2})h_n(\tau)$ so that

$$\int_{0+}^{\infty} L[f(\tau)]f(\tau)d\tau = \sum_{n=0}^{\infty} (n + \frac{1}{2})f_n^{\dagger^2} > 0$$

and

$$\int_{-\infty}^{0-} L[f(\tau)]f(\tau)d\tau = \sum_{n=-\infty}^{-1} (n + \frac{1}{2})f_n^{\dagger^2} < 0 \quad .$$

We note from integration by parts that

$$\int_{0+}^{\infty} L[f(\tau)]f(\tau)d\tau = \int_{0+}^{\infty} \{\frac{1}{4}\tau f(\tau) - \frac{d}{d\tau}\tau f'(\tau)\}f(\tau)d\tau$$

$$= \frac{1}{4}\int_{0+}^{\infty} \tau f^2(\tau)d\tau + \int_{0+}^{\infty} \tau f'(\tau)^2 d\tau \quad .$$

Similarly,

$$-\int_{-\infty}^{0-} L[f(\tau)]f(\tau)d\tau = \int_{-\infty}^{0-} \{\frac{1}{4}(-\tau)f(\tau) + \frac{d}{d\tau}\tau f'(\tau)\}f(\tau)d\tau$$

$$= \frac{1}{4}\int_{-\infty}^{0-} |\tau|f^2(\tau)d\tau + \int_{-\infty}^{0-} |\tau|f'(\tau)^2 d\tau \quad .$$

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Hence,

$$\sum_{-\infty}^{\infty} \left|n + \tfrac{1}{2}\right| f_n^{\dagger 2} = \sum_{n=0}^{\infty} (n + \tfrac{1}{2}) f_n^{\dagger 2} - \sum_{-\infty}^{-1} (n + \tfrac{1}{2}) f_n^{\dagger 2}$$

$$= \frac{1}{4} \int_{-\infty}^{\infty} |\tau| f^2(\tau)d\tau + \int_{-\infty}^{\infty} |\tau| f'(\tau)^2 d\tau \quad .$$

Dividing both sides by $\int_{-\infty}^{\infty} f^2(\tau)d\tau = \sum_{-\infty}^{\infty} f_n^{\dagger 2}$, one has

$$\bar{N}_1^\dagger = \frac{1}{4} \bar{T}_1^\dagger + \bar{B}_1 \quad , \tag{3.4}$$

proving part (1). To prove part (2), we first note the identity

$$\int_{-\infty}^{\infty} \tau f'(\tau) f(\tau) d\tau = \frac{1}{2} \int_{-\infty}^{\infty} \tau \frac{d}{d\tau} f^2(\tau) d\tau = -\frac{1}{2} \int_{-\infty}^{\infty} f^2(\tau) d\tau \quad . \tag{3.5}$$

By Schwarz's inequality,

$$\frac{1}{2} \int_{-\infty}^{\infty} f^2(\tau)d\tau \le \int_{-\infty}^{\infty} |\tau| |f'(\tau)| |f(\tau)| d\tau$$

$$\le \sqrt{\int_{-\infty}^{\infty} |\tau| f^2(\tau) d\tau} \sqrt{\int_{-\infty}^{\infty} |\tau| f'(\tau)^2 d\tau} \quad ,$$

so that

$$\frac{1}{4} \le \bar{T}_1 \bar{B}_1 \quad . \tag{3.6}$$

Substituting (3.6) into (3.4), one has the desired result.□

Remark 3.2

We see from Theorem 3.1 that when $\bar{T}_1$ becomes infinite or when $\bar{T}_1$ goes to zero, $\bar{N}_1^\dagger$ becomes infinite. The methodology therefore cannot tolerate

functions $f(\tau)$ too closely concentrated at zero or functions $f(\tau)$ too great in extent.

The inequality of Theorem 3.1 has a counterpart for the sharp transform $(f_n^{\#})_{-\infty}^{\infty}$ of equal interest. As we will see, the sharp transform is less sensitive to concentration of $f(\tau)$ at $\tau = 0$, a useful and important advantage. The following identity underlies the inequality for the sharp transform.

Theorem 3.3

Let $f(\tau) \in C_{\downarrow}^{*}(R)$. Then

$$\sum_{-\infty}^{\infty} |n| f_n^{\#2} = \int_{-\infty}^{\infty} |\tau| f^2(\tau) d\tau \quad .$$

Proof

Let $\nu(\tau) = h_0(\tau) * f(\tau)$ so that $f(\tau) = \nu'(\tau) + \frac{1}{2}\nu(\tau)$. Then

$$(3.7) \qquad \int_{-\infty}^{\infty} |\tau| f^2(\tau) d\tau = \int_{-\infty}^{\infty} |\tau| \{\nu'(\tau) + \frac{1}{2}\nu(\tau)\}^2 d\tau$$

$$= \int_{-\infty}^{\infty} |\tau| \nu'(\tau)^2 d\tau + \frac{1}{4}\int_{-\infty}^{\infty} |\tau| \nu(\tau)^2 d\tau + \int_{-\infty}^{\infty} |\tau| \nu'(\tau)\nu(\tau) d\tau \quad .$$

Since $\nu(\tau)$ is continuous at $\tau = 0$, as we will see in Theorem 4.2 of Section 4, one has $\nu_n^{\dagger} = f_n^{\#}$ for all n. Hence from Theorem 3.1, the sum of the first two terms in (3.7) is equal to $\sum_{-\infty}^{\infty} |n + \frac{1}{2}| \nu_n^{\dagger 2} = \sum_{-\infty}^{\infty} |n + \frac{1}{2}| f_n^{\# 2}$. On the other hand, the third term becomes, by an analogy to (3.5),

$$\int_{-\infty}^{\infty} |\tau| \nu'(\tau)\nu(\tau) d\tau = \int_{0}^{\infty} \tau\nu'(\tau)\nu(\tau) d\tau - \int_{-\infty}^{0} \tau\nu'(\tau)\nu(\tau) d\tau$$

$$= -\frac{1}{2}\int_{0}^{\infty} \nu^2(\tau) d\tau + \frac{1}{2}\int_{-\infty}^{0} \nu^2(\tau) d\tau$$

$$= -\frac{1}{2}\sum_{0}^{\infty} \nu_n^{+2} + \frac{1}{2}\sum_{-\infty}^{-1} \nu_n^{+2} .$$

Since $\sum_{-\infty}^{\infty} |n + \frac{1}{2}| f_n^{\#2} = \sum_{-\infty}^{\infty} |n| f_n^{\#2} + \frac{1}{2}\sum_{0}^{\infty} \nu_n^{+2} - \frac{1}{2}\sum_{-\infty}^{-1} \nu_n^{+2}$, one has from (3.7) that

$$\int_{-\infty}^{\infty} |\tau| f^2(\tau) = \sum_{-\infty}^{\infty} |n| f_n^{\#2} . \qquad \square$$

Theorem 3.3 now leads to the counterpart of Theorem 3.1 for the sharp transform $(f_n^{\#})_{-\infty}^{\infty}$. As in (3.1) for the dagger transform, we define for the sharp transform

$$(3.8) \qquad \bar{N}_1^{\#} = \sum_{-\infty}^{\infty} |n| f_n^{\#2} / \sum_{-\infty}^{\infty} f_n^{\#2} .$$

Theorem 3.4

Let $f(\tau) \in C_{\downarrow}^{*}(R)$. Then

(1) $\bar{N}_1^{\#} = \rho \bar{T}_1$ where $\rho = \sum_{-\infty}^{\infty} f_n^{+2} / \sum_{-\infty}^{\infty} f_n^{\#2}$

(2) $\bar{N}_1^{\#} \geq \frac{1}{4} \bar{T}_1$.

Proof

Part (1) is immediate from Theorem 3.3. For part (2), one has from Schwarz's inequality

$$\sum_{-\infty}^{\infty} f_n^{\#2} = \sum_{-\infty}^{\infty} (f_n^{\dagger} - f_{n-1}^{\dagger})^2 \leq 4 \sum_{-\infty}^{\infty} f_n^{+2}$$

so that $\rho \geq \frac{1}{4}$. The result then follows. $\square$

Remark 3.5

We see from Theorem 3.4 that when $\bar{T}_1$ becomes infinite, so does $\bar{N}_1^{\#}$. But in contrast to dagger transform, concentration of a function $f(\tau)$ near zero does not necessarily imply that $\bar{N}_1^{\#}$ becomes infinite. This point will be seen vividly through the following example. Let

$$f(\tau) = \{\theta e^{-\theta\tau}U(\tau)\} + \{\theta e^{\theta\tau}U(-\tau)\} , \quad \theta > 0 \quad .$$

We note that the larger θ is, the more $f(\tau)$ is concentrated near zero or, equivalently, the smaller θ is, the greater the extent of $f(\tau)$ is. One finds readily that

$$\text{(3.9)} \qquad \sum_{-\infty}^{\infty} f_n^{\dagger 2} = \theta \ ;$$

$$\text{(3.10)} \qquad \sum_{-\infty}^{\infty} |n| f_n^{\# 2} = \int_{-\infty}^{\infty} |\tau| f^2(\tau) d\tau = \frac{1}{2} \ , \text{ for all } \theta > 0 \ ;$$

$$\text{(3.11)} \qquad \sum_{-\infty}^{\infty} f_n^{\# 2} = 4\theta(\theta+1)/(2\theta+1)^2 \quad .$$

From (3.9), (3.10) and (3.11), one has

$$\text{(3.12)} \qquad \bar{T}_1 = \frac{1}{2\theta}$$

$$\text{(3.13)} \qquad \bar{N}_1^{\#} = \frac{1}{2} \frac{(2\theta+1)^2}{4\theta(\theta+1)}$$

$$\text{(3.14)} \qquad \rho = \frac{(2\theta+1)^2}{4(\theta+1)} \quad .$$

We see easily that ρ is a monotone increasing function of θ. At $\theta = 0$, $\rho = 1/4$ and therefore $\rho \geq 1/4$, as expected. We note that the inequality (2) of Theorem 3.4 is thus sharp. As $\theta \to 0$, both $\bar{T}_1$ and $\bar{N}_1^{\#}$ go to $+\infty$. But as $\theta \to +\infty$, $\bar{N}_1^{\#} \to 1/2$ while $\bar{N}_1^{\dagger} \to +\infty$ since $\bar{T}_1^{\dagger} \to 0$.

The relation $\nu_n^\dagger = f_n^\#$ with $\nu(\tau) = h_0(\tau)*f(\tau)$ played a key role in the proof of Theorem 3.3. This also enables us to evaluate $\sum_{-\infty}^{\infty} f_n^{\#^2}$ in a closed form. The relation $g_n^\dagger = nf_n^\#$ with $g(\tau) = L[f(\tau)] - f'(\tau)$ given in Theorem 4.6 of Section 4 leads to a closed form of $\sum_{-\infty}^{\infty} n^2 f_n^{\#^2}$.

<u>Theorem 3.6</u>

Let $f(\tau) \in C_\downarrow^*(R)$. Then

$$\text{(1)} \qquad \sum_{-\infty}^{\infty} f_n^{\#^2} = \int_{-\infty}^{\infty} e^{-\frac{1}{2}|\tau|} r_f(\tau)d\tau \quad \text{where} \quad r_f(\tau) = f(\tau)*f(-\tau) \quad .$$

Furthermore, if $f(\tau)$ is in $C_\downarrow^\infty(R)$,

$$\text{(2)} \qquad \sum_{-\infty}^{\infty} n^2 f_n^{\#^2} = \frac{1}{4}\int_{-\infty}^{\infty} \tau^2 f^2(\tau)d\tau + \int_{-\infty}^{\infty} \tau^2 f'(\tau)^2 d\tau \quad .$$

<u>Proof</u>

Let $\nu(\tau) = h_0(\tau)*f(\tau)$ so that $\nu_n^\dagger = f_n^\#$ as before. Then

$$\sum_{-\infty}^{\infty} f_n^{\#^2} = \sum_{-\infty}^{\infty} \nu_n^{\dagger^2} = \int_{-\infty}^{\infty} \{h_0(\tau)*f(\tau)\}^2 d\tau \quad .$$

Let $\phi(u) = \int_{-\infty}^{\infty} e^{iu\tau} f(\tau)d\tau$ and $\tilde{h}(u) = \int_{-\infty}^{\infty} e^{iu\tau} h_0(\tau)d\tau$. By the Parseval identity, one has

$$\sum_{-\infty}^{\infty} f_n^{\#^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\tilde{h}(u)|^2 |\phi(u)|^2 du$$

$$= \int_{-\infty}^{\infty} \{h_0(\tau)*h_0(-\tau)\}\{f(\tau)*f(-\tau)\}d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2}|\tau|} r_f(\tau)d\tau \quad ,$$

proving (1). To prove part (2), we let $g(\tau) = L[f(\tau) - f'(\tau)]$. Then, as given in Theorem 4.6, one has $g_n^{\dagger} = nf_n^{\#}$ for $f(\tau) \in C_{+}^{\infty}(R)$. Therefore, again by the Parseval identity,

$$\sum_{-\infty}^{\infty} n^2 f_n^{\#^2} = \int_{-\infty}^{\infty} \{L[f(\tau)] - f'(\tau)\}^2 d\tau$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} (\frac{1}{4} + u^2) |\frac{d}{du} \phi(u)|^2 du$$

$$= \frac{1}{4} \int_{-\infty}^{\infty} \tau^2 f^2(\tau) d\tau + \int_{-\infty}^{\infty} \{f(\tau) + \tau f'(\tau)\}^2 d\tau$$

$$= \frac{1}{4} \int_{-\infty}^{\infty} \tau^2 f^2(\tau) d\tau + \int_{-\infty}^{\infty} \tau^2 f'(\tau)^2 d\tau \quad . \qquad \square$$

Theorem 3.6 gives relations for the second moments of the extent of the sharp transform and the original function. Let

(3.15) $$\bar{N}_2^{\#} = \sum_{-\infty}^{\infty} n^2 f_n^{\#^2} / \sum_{-\infty}^{\infty} f_n^{\#^2} \ ;$$

(3.16) $$\bar{T}_2 = \int_{-\infty}^{\infty} \tau^2 f(\tau) d\tau / \int_{-\infty}^{\infty} f^2(\tau) d\tau \ ;$$

(3.17) $$\bar{B}_2 = \int_{-\infty}^{\infty} \tau^2 f'(\tau)^2 d\tau / \int_{-\infty}^{\infty} f^2(\tau) d\tau \quad .$$

One can easily show that

(3.18a) $$\bar{N}_2^{\#} \geq \bar{N}_1^{\#^2}$$

(3.18b) $$\bar{T}_2 \geq \bar{T}_1^2 \quad .$$

Theorem 3.7

Let $f(\tau) \in C^{\infty}_{\downarrow}(R)$.

(1) $\bar{N}_2^{\#} = \rho(\frac{1}{4}\bar{T}_2 + \bar{B}_2)$

(2) $\dfrac{\bar{N}_2^{\#}}{\bar{N}_1^{\#}} = (\frac{1}{4}\bar{T}_2 + \bar{B}_2)/\bar{T}_1$

(3) $\frac{1}{4}\bar{T}_1 \le \bar{N}_1^{\#} \le (\frac{1}{4}\bar{T}_2 + \bar{B}_2)/\bar{T}_1$

(4) $\bar{N}_2^{\#} \ge (\frac{1}{4} r + \frac{1}{r})\bar{N}_1^{\#}$ where $r = \bar{T}_2/\bar{T}_1$.

Proof

Part (1) is immediate from Theorem 3.6. From Theorem 3.4, $\bar{N}_1^{\#} = \rho\bar{T}_1$ and part (2) follows. From (3.18a),

$$\bar{N}_1^{\#} \le \frac{\bar{N}_2^{\#}}{\bar{N}_1^{\#}} = (\frac{1}{4}\bar{T}_2 + \bar{B}_2)/\bar{T} \quad .$$

$\frac{1}{4}\bar{T}_1 \le \bar{N}_1^{\#}$ is proven in Theorem 3.4, and therefore part (3) is shown. To prove part (4), we first show the inequality

$$\bar{T}_1^2 \le \bar{T}_2\bar{B}_2 \quad . \tag{3.19}$$

Let sign $(\tau) = 1$ for $\tau \ge 0$ and sign $(\tau) = -1$ for $\tau < 0$. Then

$$\int_{-\infty}^{\infty} \mathrm{sign}(\tau)f'(\tau)f(\tau)\tau^2 d\tau = \int_0^{\infty} \tau^2 f'(\tau)f(\tau)d\tau - \int_{-\infty}^{0} \tau^2 f'(\tau)f(\tau)d\tau \quad .$$

Integrating by parts, one has

$$\int_0^{\infty} \tau^2 f'(\tau)f(\tau)d\tau = \frac{1}{2}\int_0^{\infty} \tau^2 \frac{d}{d\tau} f^2(\tau)d\tau = -\int_0^{\infty} \tau f^2(\tau)d\tau \quad .$$

Similarly, $-\int_{-\infty}^{0} \tau^2 f'(\tau)f(\tau)d\tau = \int_{-\infty}^{0} \tau f^2(\tau)d\tau = -\int_{-\infty}^{0} |\tau| f^2(\tau)d\tau$, so that

$$\int_{-\infty}^{\infty} |\tau| f^2(\tau)d\tau = \int_{-\infty}^{\infty} \operatorname{sign}(\tau) f'(\tau)f(\tau)\tau^2 d\tau \quad .$$

Hence, by Schwarz's inequality,

$$\int_{-\infty}^{\infty} |\tau| f^2(\tau)d\tau \le \sqrt{\int_{-\infty}^{\infty} \tau^2 f'(\tau)^2 d\tau} \; \sqrt{\int_{-\infty}^{\infty} \tau^2 f(\tau)^2 d\tau} \quad .$$

Dividing both sides by $\int_{-\infty}^{\infty} f^2(\tau)d\tau$ and then squaring, one has (3.19). Part (4) follows from Part (2) and (3.19).□

4. Operational properties of the bilateral Laguerre transform

In principle, operational rules for the bilateral Laguerre transform can be obtained from the rules for one-sided support (cf. [A], Appendix C). One works on $f_+(\tau) = f(\tau)U(\tau)$ and $f_-(\tau) = f(\tau)U(-\tau)$ separately and then patches the results together carefully. In particular, $\int_\tau^\infty f(y)dy$ may be evaluated in that way only, when $\int_{-\infty}^\infty f(\tau)d\tau \neq 0$ and $\int_\tau^\infty f(y)dy \notin L_2(-\infty, \infty)$. The operational rules for the bilateral Laguerre transform are of separate interest, however, as we will see. They contain those for one-sided support as a special case. We also derive some new results which are of theoretical and practical value. As in Section 3, we discuss only rapidly decreasing functions $f(\tau)$ in $C_+^*(R)$. All the results are summarized concisely in a table in Appendix A.

We first extend (1.15) and (1.16).

P4.1

Let $f(\tau) \in C_+^*$. Then

$$g(\tau) = f(-\tau) \cdot \Longleftrightarrow g_n^\# = f_{-n}^\# \; ; \quad g_n^\dagger = -f_{-(n+1)}^\dagger \quad .$$

Proof

Since $\gamma_B(s) = \phi_B(-s) = -\sum_{-\infty}^{\infty} f_n^\dagger\left(\frac{s+\frac{1}{2}}{s-\frac{1}{2}}\right)^n \frac{1}{s-\frac{1}{2}}$, one has

$$T_g^\#(u) = \gamma_B\left(\frac{1}{2}\,\frac{1+u}{1-u}\right) = \left(1 - \frac{1}{u}\right) \sum_{-\infty}^{\infty} f_n^\dagger u^{-n} = T_f^\#(u^{-1}) \quad .$$

Similarly, $T_g^\dagger(u) = \frac{1}{1-u}\, T_g^\#(u) = -\frac{1}{u}\, T_f^\dagger(u^{-1})$. □

When a function is convolved with one of the building block functions $h_m(\tau)$, the dagger coefficients of the resulting function become the sharp coefficients of the original function shifted by -m. We see this next. For a sequence $(a_n)_{-\infty}^{\infty}$, the first difference will be denoted by $\Delta a_n = a_n - a_{n-1}$,

and the second difference by $\Delta^2 a_n = \Delta a_n - \Delta a_{n-1} = a_n - 2a_{n-1} + a_{n-2}$.

Theorem 4.2

Let $f(\tau) \in C^*_\downarrow(R)$. For all integers m, one has

$$g(\tau) = h_m(\tau)*f(\tau) \Longleftrightarrow g_n^\# = \Delta f^\#_{n-m} \;;\quad g_n^\dagger = f^\dagger_{n-m} \quad .$$

Proof

It is easy to see that $T^\#_{h_m}(u) = (1-u)u^m$ for all integers m. Then $T^\#_g(u) = (1-u)u^m T^\#_f(u)$ and the result follows.□

Of particular interest is the case m = 0, which provides a closed form of $\sum_{-\infty}^{\infty} f_n^{\#2}$, as given in Theorem 3.6. The next corollary is immediate from Theorem 4.2.

Corollary 4.3

Let $f(\tau) \in C^*_\downarrow(R)$. Then

(1) $g(\tau) = f(\tau) - h_0(\tau)*f(\tau) \Longleftrightarrow g_n^\# = f^\#_{n-1} \;;\quad g_n^\dagger = f^\dagger_{n-1}$

(2) $g(\tau) = f(\tau) + h_{-1}(\tau)*f(\tau) \Longleftrightarrow g_n^\# = f^\#_{n+1} \;;\quad g_n^\dagger = f^\dagger_{n+1}$.

The bilateral Laguerre transform of a derivative of a function is a straightforward extension of the one-sided case with slight modification.

Theorem 4.4

Let $f(\tau) \in C^*_\downarrow(R)$. Then

$$g(\tau) = \frac{d}{d\tau} f(\tau) \Longleftrightarrow \begin{cases} g_n^\# = \frac{1}{2}(f_n^\dagger + f^\dagger_{n-1}) - \delta_{n,0}\{f(0+) - f(0-)\} \\ g_n^\dagger = \begin{cases} \frac{1}{2} f_n^\dagger - \sum_{m=n}^{\infty} f_m^\dagger \,, & n \ge 0 \\ -\frac{1}{2} f_n^\dagger + \sum_{-\infty}^{n} f_m^\dagger \,, & n < 0 \end{cases} \end{cases}$$

Proof

This is immediate from the corresponding rule for one-sided support ([A], Appendix C).□

We have seen in Theorem 4.2 that $g_n^{\dagger} = f_n^{\#}$ with $g(\tau) = h_0(\tau)*f(\tau)$. The next theorem gives the reversed relation of this, i.e., functions $g(\tau)$ and $f(\tau)$ related by $g_n^{\#} = f_n^{\dagger}$.

Theorem 4.5

Let $f(\tau) \in C_{\downarrow}^{*}(R)$. Then

$$g(\tau) = \frac{d}{d\tau} f(\tau) + \frac{1}{2} f(\tau) \iff \begin{cases} g_n^{\#} = f_n^{\dagger} - \delta_{n,0}\{f(0+) - f(0-)\} \\ g_n^{\dagger} = -\sum_{n+1}^{\infty} \left[f_m^{\dagger} - \delta_{m,0}\{f(0+) - f(0-)\}\right] . \end{cases}$$

Proof

From Theorem 4.4,

$$\begin{aligned} g_n^{\#} &= \tfrac{1}{2}(f_n^{\dagger} + f_{n-1}^{\dagger}) - \delta_{n,0}\{f(0+) - f(0-)\} + \tfrac{1}{2} f_n^{\#} \\ &= f_n^{\dagger} - \delta_{n,0}\{f(0+) - f(0-)\} \quad . \quad \square \end{aligned}$$

We recall the operator $L - [\frac{1}{4}\tau - \frac{d}{d\tau}\tau\frac{d}{d\tau}]$ given in (2.5b). For functions $f(x) \in C_{\downarrow}^{*}(R)$, one has immediately from (2.6) that

$$(4.1) \qquad g(\tau) = L^{(r)}[f(\tau)] \iff g_n^{\#} = \Delta\{(n+1/2)^r f_n^{\dagger}\} \; ; \quad g_n^{\dagger} = (n+1/2)^r f_n^{\dagger}$$

for $r = 0,1,2,\ldots$. This leads to the following theorem which is of theoretical value, providing a closed form of $\sum_{-\infty}^{\infty} n^2 {f_n^{\#}}^2$ given in Theorem 3.6.

Theorem 4.6

Let $f(\tau) \in C_{\downarrow}^{*}(R)$. Then

$$g(\tau) = L[f(\tau)] - \frac{d}{d\tau} f(\tau) \iff \begin{cases} g_n^{\#} = nf_n^{\#} + \delta_{n,0}\{f(0+) - f(0-)\} \\ \\ g_n^{\dagger} = - \sum\limits_{n+1}^{\infty} [mf_m^{\#} + \delta_{m,0}\{f(0+) - f(0-)\}]. \end{cases}$$

Proof

From (4.1) with r = 1 and Theorem 4.4,

$$g_n^{\#} = (n + 1/2)f_n^{\dagger} - (n - 1/2)f_{n-1}^{\dagger} - \frac{1}{2}(f_n^{+} + f_{n-1}^{+}) + \delta_{n,0}\{f(0+) - f(0-)\}$$

$$= n\Delta f_n^{\dagger} + \delta_{n,0}\{f(0+) - f(0-)\} = nf_n^{\#} + \delta_{n,0}\{f(0+) - f(0-)\} . \quad \square$$

The bilateral Laguerre transform of $\tau f(\tau)$ is exactly the same as for the one-sided case. The proof is straightforward and omitted.

Theorem 4.7

Let $f(\tau) \in C_{\downarrow}^{*}(R)$. Then

$$g(\tau) = \tau f(\tau) \iff g_n^{\#} = - \Delta^2[(n+1)f_{n+1}^{\#}] \ ; \quad g_n^{\dagger} = - \Delta[(n+1)f_{n+1}^{\#}] \quad .$$

From Appendix A of [A], one finds that

(4.2) $$\sum_{-\infty}^{\infty} h_n(\tau)u^n = \frac{1}{1-u} \exp[- \frac{\tau}{2} \frac{1+u}{1-u}]$$

with the understanding that Eq. (4.2) is valid for $|u| < 1$ when $\tau \geq 0$ and for $|u| > 1$ when $\tau < 0$. This leads to the operational rule for shifting given in Theorem 4.9. A preliminary remark is needed.

Remark 4.8

Even though $(h_n(T))_{-\infty}^{\infty} \notin \ell_2$, one sees easily that

(4.3a) $$(\Delta h_n(T))_{-\infty}^{\infty} \in \ell_2$$

(4.3b) $\sum_{-\infty}^{\infty} (\Delta h_n(T))^2 = 1$

(4.3c) $|\Delta h_n(T)| < 1$ for all integers n, $T \neq 0$.

These statements follow from

(4.4) $[h_0(\tau - T)]_n^{\dagger} = \Delta h_n(T)$,

as the reader will verify, and

(4.5) $\int_{-\infty}^{\infty} h_0^2(\tau - T)d\tau = \int_{-\infty}^{\infty} h_0^2(\tau)d\tau = 1$.

<u>Theorem 4.9</u>

Let $f(\tau) \in C_{\downarrow}^{*}(R)$. Then

$$g(\tau) = f(\tau - T) \iff g_n^{\#} = \sum_{-\infty}^{\infty} f_{n-m}^{\#}\Delta h_m(T) \ ; \quad g_n^{\dagger} = -\sum_{n+1}^{\infty} g_n^{\#} \ .$$

<u>Proof</u>

Clearly $\gamma_B(s) = e^{-sT}\phi_B(s)$ so that

$$T_g^{\#}(u) = (1-u)\frac{1}{1-u}\exp\{-\frac{T}{2}\frac{1+u}{1-u}\}T_f^{\#}(u)$$

$$= (1-u)\sum_{-\infty}^{\infty} h_n(T)u^n T_f^{\#}(u)$$

$$= \sum_{-\infty}^{\infty} (\Delta h_n(T))u^n \cdot T_f^{\#}(u) \ ,$$

and the result follows. □

5. Applications

In applied probability theory and statistics, the prevalence of a random variable $S_k = \sum_{j=1}^{k} \xi_j$, where ξ_j are i.i.d. random variables, is familiar. One also often encounters the ergodic green density $g(\tau) = \delta(\tau) + \sum_{k=1}^{\infty} a^{(k)}(\tau)$ in expressions for the ergodic distributions of homogeneous processes modified by boundaries [4], [5]. Here $a^{(k)}(\tau)$ denotes the k-fold convolution of a two-sided p.d.f. $a(\tau)$ with itself. The bilateral Laguerre transform method provides an algorithmic basis for calculating multiple convolutions of functions with two-sided support and thus enables one to evaluate the distribution of S_k or the ergodic green density $g(\tau)$ numerically. This in turn is the key to numerical evaluation of a variety of results in applied probability that have been available only formally.

In the expression $S_K = \sum_{j=1}^{K} \xi_j$, when ξ_j are centered, the central limit theorem says that S_K with suitable normalization converges to a standard normal random variable in distribution. In some applications, the question of importance is "How fast is the convergence to normality?". This can be also answered by the method directly via computation of the required multiple convolutions.

In this section we discuss the algorithmic procedure for the calculation of multiple two-sided convolutions and present a numerical example. This procedure is then used to quantify the Lindley process. For the process in queueing contexts, the ergodic waiting time distributions for M/G/1 and G/G/1 systems are evaluated thereby. All computations were done on a DEC 10 computer in a time-sharing mode using APL as the programming language. The DEC 10 APL implementation is a double precision system, which uses a

precision of 18 decimal digits. Relevant formulas were usually coded in a straightforward way, with no attempt made to optimize the subroutines for speed or accuracy. In spite of this, the results displayed here were typically obtained with CPU times in seconds to at most a few minutes and with no evidence of numerical problems.

Example 1: Multiple two-sided convolutions

Two-sided convolutions are usually handled by Fourier transform methods. There does not appear to be any significant literature devoted to numerical evaluation of two-sided convolutions. When the function being convolved is rapidly decreasing, i.e., in the class $C_{+}^{*}(R)$ defined in Section 3, our method yields a fast and accurate algorithm. To generate multiple convolution, we make use of (1.13) which states that $T^{\#}_{f*g}(u) = T^{\#}_{f}(u)T^{\#}_{g}(u)$. Hence $(f^{\#}_{n}) = (a^{\#}_{n})^{(m)}$ where $f(t) = a^{(m)}(t)$, and one has the following algorithm:

I. Representing $a(t)$ as $a(t) = a_{+}(t) + a_{-}(t)$ as in Section 1, generate or store in the computer the coefficients $(a^{\dagger}_{n+})_{0}^{N}$ and $(a^{\dagger}_{n-})_{0}^{N}$, usually obtained analytically as in [A].

II. Convert $(a^{\dagger}_{n+})$ and $(a^{\dagger}_{n-})$ to $(a^{\#}_{n+})$ and $(a^{\#}_{n-})$, respectively. This corresponds to a simple differencing operation on each set of coefficients, since $T^{\#}(u) = (1-u)T^{\dagger}_{f}(u)$.

III. Obtain $(a^{\#}_{n})_{-N}^{N}$ by setting $a^{\#}_{n} = a^{\#}_{n+}$ for $n > 0$, $a^{\#}_{0} = a^{\#}_{0+} + a^{\#}_{0-}$, and $a^{\#}_{n} = a^{\#}_{(-n)-}$ for $n < 0$, as in (1.19).

IV. Perform m-fold discrete convolution on $(a^{\#}_{n})_{-N}^{N}$. Retain only 2N+1 terms, centered at $n = 0$, in each convolution. The result is $(f^{\#}_{n})_{-N}^{N}$.

V. Convert $(f_n^{\#})$ to $(f_n^{\dagger})$ by (1.18). This is the inverse of the differencing operation, i.e., summation.

VI. One has $f_{n+}^{\dagger} = f_n^{\dagger}$ for $n \geq 0$. Hence, sum the series $\sum_{n=0}^{N} f_n^{\dagger}\ell_n(t)$ to get $f_+(t)$. Similarly, one has $f_{n-}^{\dagger} = -f_{-n-1}^{\dagger}$ for $n \leq 0$, by virtue of (1.4d). Hence, sum the series $-\sum_{n=0}^{N} f_{-n-1}^{\dagger}\ell_n(-t)$ to get $f_-(t)$.

This procedure is only slightly more complicated than that for the one-sided convolutions given in [A], but the CPU requirements for the new procedure is increased approximately by a factor of 2. When a function $a(t)$ is symmetric about zero so that $a_+(t) = a_-(-t)$, however, one can evaluate multiple convolutions by working only on one side, as the reader will verify. Then the CPU requirements will be almost the same as for the one-sided convolutions.

We illustrate the procedure with $a(t) = \{2e^{-2t}U(t)\}*\{e^tU(-t)\}$. $(a_{n+}^{\#})$ and $(a_{n-}^{\#})$ are available analytically from [A], Appendix B. Fig. 5.1 displays $a^{(m)}(t)$ for $1 \leq m \leq 7$ and $-10 \leq t \leq 10$. Two hundred terms each for $a_+(t)$ and $a_-(t)$ provided an accuracy of at least 12 digits after the decimal point uniformly over the interval $-10 \leq t \leq 10$.

The procedure is also applied in [11] to numerical evaluation of multiple convolutions of the Logistic variate with the p.d.f. $a(t) = e^{-t}/(1 + e^{-t})^2$, $-\infty < t < \infty$. The Laguerre transform approach appeared to be more systematic and efficient than an existing method [2] which relied on the special analytic feature of the Logistic variate. The Laguerre method also evaluated multiple convolutions of the folded Logistic variate as a by-product, which the existing method could not provide.

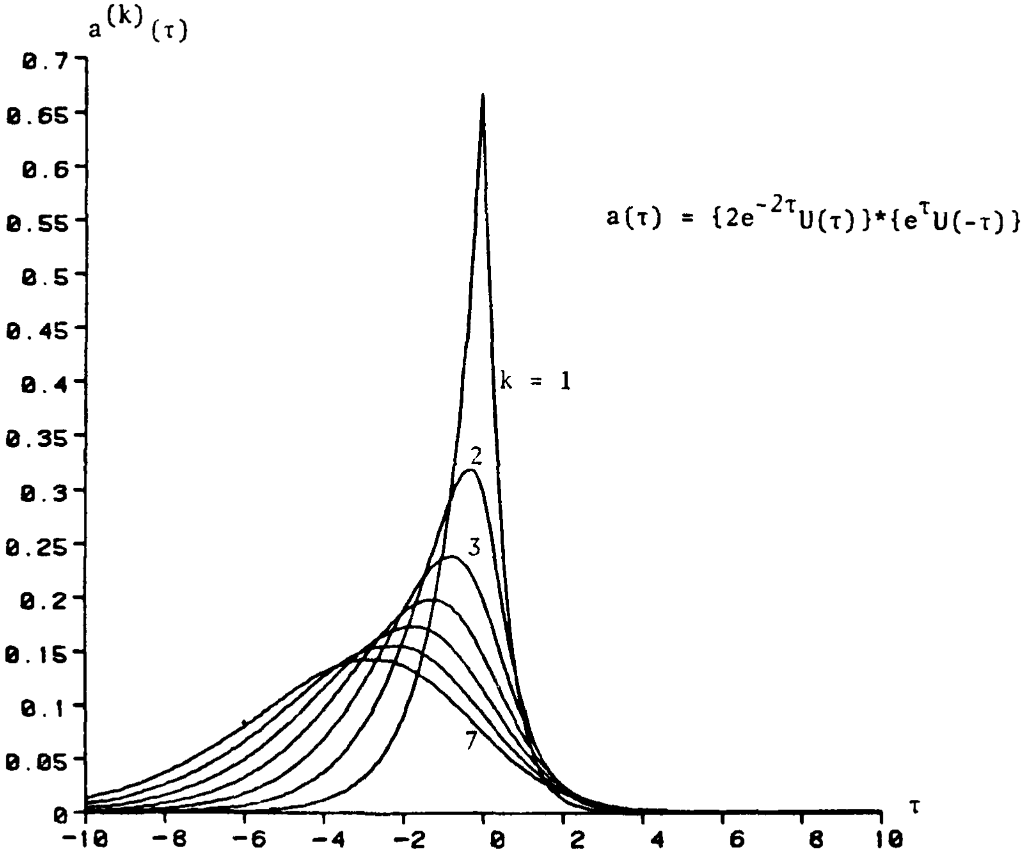

Fig. 5.1. Multiple two-sided convolutions $a^{(k)}(\tau)$

Example 2: The Lindley process

Consider the Lindley process [6], i.e., a random walk on the nonnegative half line defined by

$$(5.1) \qquad W_{k+1} = [W_k + \xi_{k+1}]^+ , \quad k = 0,1,2,\ldots$$

where $[X]^+ = \max\{0,X\}$. W_0 is a random variable with known distribution and ξ_k are i.i.d. and independent of W_0. We assume that the common distribution of ξ_k is absolutely continuous with density function $a(x) \in L_2(-\infty, \infty)$. We see that the distribution of W_{k+1} is the convolution of the distributions of W_k and ξ_{k+1}, modified by the $[\;]^+$ operation. The effect of this operation is to "sweep up" the probability to the left of the origin into a mass point at the origin. Let

$$(5.2a) \qquad E_k = P[W_k = 0]$$

$$(5.2b) \qquad F_k(x) = P[W_k \le x] = \begin{cases} 0 & x < 0 \\ E_k + \int_0^x f_k(y)dy & x \ge 0 \end{cases} .$$

If we define $W^H_{k+1} = W_k + \xi_{k+1}$, then the density of W^H_{k+1} is given by

$$(5.3) \qquad f^H_{k+1}(x) = E_k a(x) + f_k(x) * a(x) \quad .$$

Hence if E_k and the Laguerre sharp coefficients $(a^\#_n)^\infty_{-\infty}$ and $(f^\#_n(k))^\infty_0$ are known, one has

$$(5.4) \qquad f^{H\#}_n(k+1) = E_k a^\#_n + \sum_{m=0}^{\infty} a^\#_{n-m} f^\#_m(k) \; , \quad -\infty < n < \infty \quad .$$

Obviously, $f_{k+1}(x) = f^{H}_{k+1}(x)U(x)$. Hence, since $\sum_{-\infty}^{\infty} f^{\#}_n = 0$ (cf. (1.17)),

$$(5.5) \qquad f^{\#}_0(k+1) = \sum_{n=-\infty}^{0} f^{H\#}_n(k+1) \ ; \quad f^{\#}_n(k+1) = f^{H\#}_n(k+1) \ , \quad n \geq 1 \ .$$

Finally, $E_{k+1} = 1 - \int_0^{\infty} f_{k+1}(y)dy = 1 - 2 \sum_{n=0}^{\infty} (-1)^n f^{\dagger}_n(k+1)$ since $\int_0^{\infty} \ell_n(x)dx = (-1)^n 2$ (cf. [A], Appendix A). This leads to

$$(5.6) \qquad E_{k+1} = 1 + 2 \sum_{n=0}^{\infty} f^{\#}_{2n+1}(k+1) \ .$$

Hence, if E_0, $f_0(x)$ and $a(x)$ are known, we have an iterative schema for calculating E_k and $f_k(x)$ via (5.4), (5.5) and (5.6).

We illustrate the procedure in a queueing context. A stream of customers arrive at a single-server queue at a sequence of arrival epochs τ_0, τ_1, etc., the k-th customer arriving at epoch τ_k. The interarrival times $T_k = \tau_{k+1} - \tau_k$ are i.i.d. with common distribution $T(x)$. The service times S_k required by the k-th customer form a separate sequence of i.i.d. random variables with common distribution $S(x)$, where S_k and T_k are independent. If W_k is the time the k-th customer must wait in queue for service, then one has [6]

$$(5.7) \qquad W_{k+1} = [W_k + \xi_{k+1}]^+ \ ; \quad \xi_{k+1} = S_k - T_k \ , \quad k = 0,1,2,\ldots$$

This process is called the Lindley waiting-time process. Clearly, the common density of ξ_k is given by

$$(5.8) \qquad a(x) = \{s(x)U(x)\} * \{t(-x)U(-x)\}$$

where $s(x)$ and $t(x)$ are the density functions for the service time and

interarrival time variables, respectively. When the system starts with an empty queue, one has $E_0 = 1$ and $f_0(x) = 0$.

The iterative schema (5.4), (5.5) and (5.6) was first tested in the M/M/1 setting for which the limiting distribution of W_k as k goes to infinity is analytically available [7]. For the traffic intensity $\rho = 0.5$, an accuracy of 7 digits after the decimal point was attained at $k = 50$, while for $\rho = 0.67$, $k = 128$ was needed to attain the same accuracy level.

The procedure was then applied to the M/G/1 queueing system with $s(x) = xe^{-x^2/2}$, the Rayleigh distribution and $t(x) = \frac{1}{2} e^{-\frac{1}{2}x}$. The survival functions of W_k for various values of k are displayed in Fig. 5.2. This shift from $f_k(x)$ to $\bar{F}_k(x) = \int_x^\infty f_k(y)dy$ is immediate using an operational property of the Laguerre transform [A], Appendix C. The results were compared with the numerical results obtained from the Khinchin-Pollaczek formula [7], again via Laguerre transform based calculation. The difference between the two results was bounded by 10^{-8} at $k = 100$ and by 10^{-12} at $k = 200$.

Fig. 5.3 displays similar results for the G/G/1 queueing system with $s(x) = \sqrt{2/\pi}\, e^{-\frac{1}{2}x^2}$, the folded normal distribution (Chi distribution with 1 d.f.) and $t(x) = xe^{-\frac{1}{2}x^2}$, the Rayleigh distribution. For this G/G/1 setting, no analytical results in the real domain are available.

Even though the traffic intensity for the M/G/1 example ($\rho = 0.63$) is less than that for the G/G/1 example ($\rho = 0.64$), the convergence in M/G/1 is much slower. We note that $\bar{F}_k(0+) \to \rho$ as $k \to \infty$ in the M/G/1 example, while this does not hold in the G/G/1 example.

The Laguerre dagger coefficients of the folded normal density were generated by an efficient recurrence formula developed in [10]. Those for the Rayleigh density were then derived using an operational property of

the Laguerre transform [A], Appendix C. In the computations 250 terms were used in each of the Laguerre series (1.1) for s(x) and t(x).

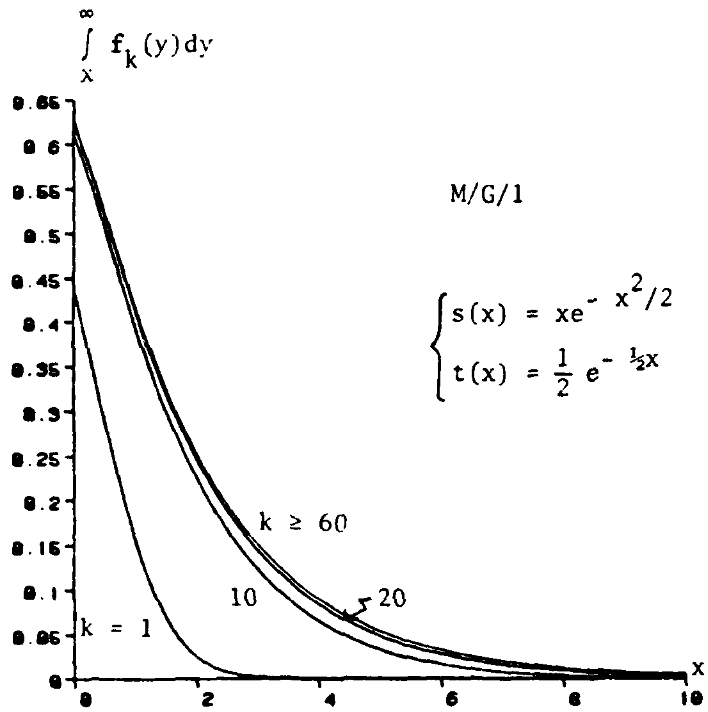

Fig. 5.2. The survival function of the waiting time of the k-th customer

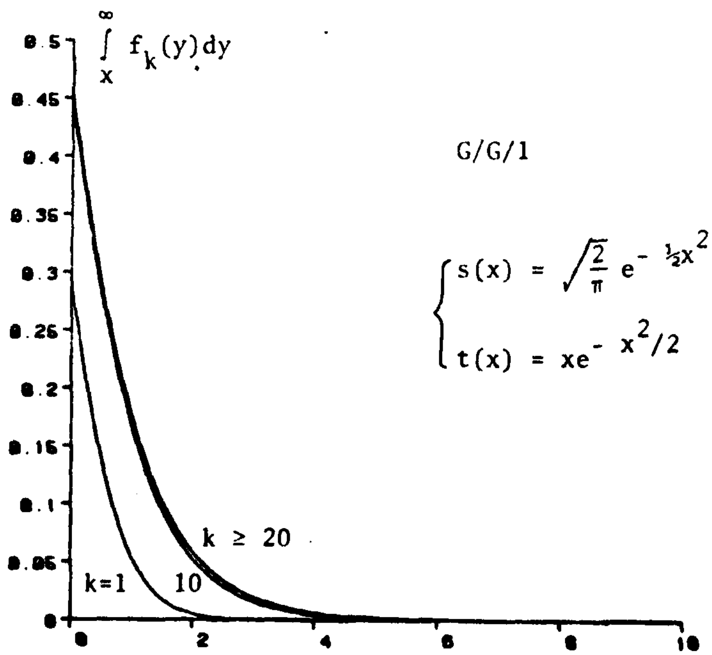

Fig. 5.3. The survival function of the waiting time of the k-th customer

APPENDIX A

Operational Properties of Bilateral Laguerre Transforms

$f(\tau)$	$\varphi_B(s)$	$T_f^*(u)$	$T_f^\dagger(u)$	f_n^*	$f_n^\dagger$
$f(\tau)$	$\int_{-\infty}^{\infty} e^{-s\tau} f(\tau)\,d\tau$	$\sum_{-\infty}^{\infty} f_n^* u^n = \varphi_B\left(\frac{1+u}{2(1-u)}\right)$	$\sum_{-\infty}^{\infty} f_n^\dagger u^n = \frac{1}{1-u} T_f^*(u)$	$\Delta f_n^\dagger$	$\int_{-\infty}^{\infty} f(\tau)\, \ell_n(\tau)\,d\tau$
$f(\tau) * g(\tau)$	$\varphi_B(s)\, \gamma_B(s)$	$T_f^*(u)\, T_g^*(u)$	$(1-u)\, T_f^\dagger(u)\, T_g^\dagger(u)$	$\sum_{m=-\infty}^{\infty} f_{n-m}^*\, g_m^*$	$-\sum_{k=n+1}^{\infty} \sum_{m=-\infty}^{\infty} f_{k-m}^*\, g_m^*$
$f(-\tau)$	$\varphi_B(-s)$	$T_f^*(u^{-1})$	$-\frac{1}{u} T_f^\dagger(u^{-1})$	f_{-n}^*	$-f_{-(n+1)}^\dagger$
$\ell_m(\tau) * f(\tau)$	$\frac{1}{(s+\frac{1}{2})}\left(\frac{s-\frac{1}{2}}{s+\frac{1}{2}}\right)^m \varphi_B(s)$	$(1-u)\, u^m\, T_f^*(u)$	$(1-u)\, u^m\, T_f^\dagger(u)$	Δf_{n-m}^*	f_{n-m}^*
$f(\tau) - \ell_0(\tau) * f(\tau)$	$\left(\frac{s-\frac{1}{2}}{s+\frac{1}{2}}\right) \varphi_B(s)$	$u\, T_f^*(u)$	$u\, T_f^\dagger(u)$	f_{n-1}^*	$f_{n-1}^\dagger$
$f(\tau) + \ell_{-1}(\tau) * f(\tau)$	$\left(\frac{s+\frac{1}{2}}{s-\frac{1}{2}}\right) \varphi_B(s)$	$\frac{1}{u} T_f^*(u)$	$\frac{1}{u} T_f^\dagger(u)$	f_{n+1}^*	$f_{n+1}^\dagger$
* $\frac{d}{d\tau} f(\tau)$	$s\, \varphi_B(s)$	$\frac{1}{2}\left(\frac{1+u}{1-u}\right) T_f^*(u)$	$\frac{1}{2}\left(\frac{1+u}{1-u}\right) T_f^\dagger(u)$	$\frac{1}{2}\left(f_n^\dagger + f_{n-1}^\dagger\right)$	$\frac{1}{2} f_n^\dagger - \sum_{m=n}^{\infty} f_m^\dagger$
* $\frac{d}{d\tau} f(\tau) + \frac{1}{2} f(\tau)$	$(s+\frac{1}{2})\, \varphi_B(s)$	$\frac{1}{1-u}\, T_f^*(u)$	$\frac{1}{1-u}\, T_f^\dagger(u)$	$f_n^\dagger$	$-\sum_{m=n+1}^{\infty} f_m^\dagger$
* $L[f(\tau)] - \frac{d}{d\tau} f(\tau)$	$(s^2 - \frac{1}{4}) \frac{d}{ds} \varphi_B(s)$	$u \frac{d}{du} T_f^*(u)$	$u \frac{d}{du} T_f^\dagger(u) - \frac{u}{1-u} T_f^\dagger(u)$	$n f_n^*$	$-\sum_{m=n+1}^{\infty} m f_m^*$
$\tau f(\tau)$	$-\frac{d}{ds} \varphi_B(s)$	$-(1-u)^2 \frac{d}{du} T_f^*(u)$	$(1-u)\, T_f^\dagger(u) - (1-u)^2 \frac{d}{du} T_f^\dagger(u)$	$-\Delta^2\left[(n+1) f_{n+1}^*\right]$	$-\Delta\left[(n+1) f_{n+1}^*\right]$
$f(\tau - T)$	$e^{-sT} \varphi_B(s)$	$\exp\left\{-\frac{T}{2} \frac{1+u}{1-u}\right\} T_f^*(u)$	$\exp\left\{-\frac{T}{2} \frac{1+u}{1-u}\right\} T_f^\dagger(u)$	$\sum_{-\infty}^{\infty} f_{n-m}^*\, \Delta \ell_m(T)$	$-\sum_{k=n+1}^{\infty} \sum_{m=-\infty}^{\infty} f_{k-m}^*\, \Delta \ell_m(T)$

Note: $f(\tau)$, $g(\tau) \in C_+^\infty(R)$ is assumed. When $f(\tau) \in C_+^*(R)$, the operational rules above are unchanged except for those functions marked with *. The required modifications are given in Section 4.

APPENDIX B

Some useful identities for the Laguerre coefficients $(f_n^\dagger)$ and $(f_n^\#)$

As before, $f(\tau) \in C_+^*(R)$ is assumed.

(1) $f(\tau) = \sum_{-\infty}^{\infty} f_n^\dagger h_n(\tau)$

(2) $f_n^\dagger = \int_{-\infty}^{\infty} f(\tau) h_n(\tau) d\tau$

(3) $\sum_{-\infty}^{\infty} {f_n^\dagger}^2 = \int_{-\infty}^{\infty} f^2(\tau) d\tau$

(4) $f_n^\# = f_n^\dagger - f_{n-1}^\dagger \; ; \quad f_n^\dagger = \sum_{-\infty}^{n} f_m^\# = - \sum_{n+1}^{\infty} f_m^\#$

(5) $T_f^\#(u) = \sum_{-\infty}^{\infty} f_n^\# u^n = \phi_B(\frac{1}{2}\,\frac{1+u}{1-u}) \; ; \quad \phi_B(s) = \int_{-\infty}^{\infty} e^{-s\tau} f(\tau) d\tau$

(6) $T_f^\#(e^{i\theta}) = \phi_B(\frac{i}{2} \cot \frac{\theta}{2})$

(7) $\sum_{-\infty}^{\infty} f_n^\dagger = - \sum_{-\infty}^{\infty} n f_n^\# = f(0+) - f(0-)$

(8) $\sum_{-\infty}^{\infty} f_n^\# = 0$

(9) $2 \sum_{-\infty}^{\infty} (-1)^n f_n^\dagger = \sum_{-\infty}^{\infty} (-1)^n f_n^\# = \int_{-\infty}^{\infty} f(\tau) d\tau$

(10) $\sum_{-\infty}^{\infty} (-1)^n n f_n^\# = \frac{1}{4} \int_{-\infty}^{\infty} \tau f(\tau) d\tau$

(11) $\sum_{-\infty}^{\infty} (-1)^n n^2 f_n^\# = \frac{1}{16} \int_{-\infty}^{\infty} \tau^2 f(\tau) d\tau$

$$* \ (12) \quad \sum_{-\infty}^{\infty} f_n^{\#^2} = \int_{-\infty}^{\infty} e^{-\frac{1}{2}|\tau|} r_f(\tau)d\tau \ ; \quad r_f(\tau) = f(\tau)*f(-\tau)$$

$$* \ (13) \quad \sum_{-\infty}^{\infty} |n| f_n^{\#^2} = \int_{-\infty}^{\infty} |\tau| f^2(\tau)d\tau$$

$$* \ (14) \quad \sum_{-\infty}^{\infty} n^2 f_n^{\#^2} = \frac{1}{4}\int_{-\infty}^{\infty} \tau^2 f^2(\tau)d\tau + \int_{-\infty}^{\infty} \tau^2 f'(\tau)^2 d\tau$$

* For (12), (13) and (14), $f(\tau) \in C_{\downarrow}^{\infty}(R)$ is assumed.

References

[A] Keilson, J. and Nunn, W., "Laguerre Transformation as a Tool for the Numerical Solution of Integral Equations of Convolution Type," Applied Mathematics and Computation, Vol. 5, pp. 313-359, 1979.

[1] Dym, H. and McKean, H. P., Fourier Series and Integrals, Academic Press, New York, 1972.

[2] Goel, P. K., "On the Distribution of Standardized Mean of Samples from the Logistic Population," Sankhyā, Vol. 37, Series B. Part 2, pp. 165-172.

[3] Keilson, J., Green's Function Methods in Probability Theory, Charles Griffin and Company, Ltd., 1965.

[4] Keilson, J., "An Alternative to Wiener-Hopf Methods for the Study of Bounded Processes," Journal of Applied Probability, Vol. 1, No. 1, June 1964.

[5] Keilson, J., "The Role of Green's Functions in Congestion Theory," Symposium on Congestion Theory, University of North Carolina Press, 1965.

[6] Kingman, J.F.C., "The Heavy Traffic Approximation in the Theory of Queues," Symposium on Congestion Theory, University of North Carolina Press, 1965.

[7] Kleinrock, L., Queueing Systems, Vol. I, John Wiley & Sons, 1975.

[8] Krylov, V. I. and Skoblya, N. S., A Handbook of Methods of Approximate Fourier Transformation and Inversion of the Laplace Transformation, Mir Publishers, Moscow, 1977.

[9] Levinson, N. and Redheffer, R. M., Complex Variables, Holden-Day, Inc., 1970.

[10] Sumita, U., "Numerical Evaluation of Multiple Convolutions, Survival Functions, and Renewal Functions for the One-Sided Normal Distribution and the Rayleigh Distribution via the Laguerre Transformation," Working Paper Series No. 7912, Graduate School of Management, University of Rochester, 1979.

[11] Sumita, U., "On Sums of Independent Logistic and Folded Logistic Variants," Working Paper Series No. 8001, Graduate School of Management, University of Rochester, 1980.

[12] Weeks, W. T., "Numerical Inversion of Laplace Transforms Using Laguerre Functions," Journal of the Association for Computing Machinery, Vol. 13, No. 3, pp. 419-426, 1966.

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Coile, Russell C., "Error Detection in Computerized Information Retrieval Data Bases," July, 1977, 13 pp. (Presented at the Sixth Cranfield International Conference on Mechanized Information Storage and Retrieval Systems, Cranfield Institute of Technology, Cranfield, Bedford, England, 26-29 July 1977), AD A043 580

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Mahoney, Robert B., Jr., "European Perceptions and East-West Competition," 96 pp., July 1977 (Prepared for presentation at the annual meeting of the International Studies Association, St. Louis, Mo., March, 1977), AD A043 661

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Sawyer, Ronald, "The Independent Field Assignment: One Man's View," August 1977, 25 pp.

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Holen, Arlene, "Effects of Unemployment Insurance Entitlement on Duration and Job Search Outcome," August 1977, 6 pp., (Reprinted from Industrial and Labor Relations Review, Vol., 30, No. 4, Jul 1977)

PP 193
Horowitz, Stanley A., "A Model of Unemployment Insurance and the Work Test," August 1977, 7 pp. (Reprinted from Industrial and Labor Relations Review, Vol. 30, No. 40, Jul 1977)

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Classen, Kathleen P., "The Effects of Unemployment Insurance on the Duration of Unemployment and Subsequent Earnings," August 1977, 7 pp. (Reprinted from Industrial and Labor Relations Review, Vol. 30, No. 40, Jul 1977)

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Brechling, Frank, "Unemployment Insurance Taxes and Labor Turnover: Summary of Theoretical Findings," 12 pp. (Reprinted from Industrial and Labor Relations Review, Vol. 30, No. 40, Jul 1977)

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PP 197
Wells, Anthony R., "The Centre for Naval Analyses," 14 pp., Dec 1977, AD A049 107

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Classen, Kathleen P., "The Distributional Effects of Unemployment Insurance," 25 pp., Sept. 1977 (*Presented at a Hoover Institution Conference on* Income Distribution, Oct 7-8, 1977), AD A054 423

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Durch, William J., "Revolution From A F.A.R. – The Cuban Armed Forces in Africa and the Middle East," Sep 1977, 16 pp., AD A046 268

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Powers, Bruce F., "The United States Navy," 40 pp. Dec 1977 (Published as a chapter in The U.S. War Machine by Salamander Books, England, 1978), AD A049 108

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Durch, William J., "The Cuban Military in Africa and The Middle East: From Algeria to Angola," Sep 1977, 67 pp., AD A045 675

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Feldman, Paul, "Why Regulation Doesn't Work," (Reprinted from Technological Change and Welfare in the Regulated Industries and Review of Social Economy, Vol. XXIX, March, 1971, No. 1.) Sep 1977, 8 pp.

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Feldman, Paul, "Efficiency, Distribution, and the Role of Government in a Market Economy," (Reprinted from The Journal of Political Economy, Vol. 79, No. 3, May/June 1971.) Sep 1977, 19 pp., AD A045 675

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Wells, Anthony R., "The 1967 June War: Soviet Naval Diplomacy and The Sixth Fleet – A Reappraisal," Oct 1977, 36 pp., AD A047 236

PP 205
Coile, Russell C., "A Bibliometric Examination of the Square Root Theory of Scientific Publication Productivity," (Presented at the annual meeting of the American Society for Information Science, Chicago, Illinios, 29 September 1977.) Oct 1977, 6 pp., AD A047 237

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McConnell, James M., "Strategy and Missions of the Soviet Navy in the Year 2000," 48 pp., Nov 1977 (Presented at a Conference on Problems of Sea Power as we Approach the 21st Century, sponsored by the American Enterprise Institute for Public Policy Research, 6 October 1977, and subsequently published in a collection of papers by the Institute), AD A047 244

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Goldberg, Lawrence, "Cost-Effectiveness of Potential Federal Policies Affecting Research & Development Expenditures in the Auto, Steel and Food Industries," 36 pp., Oct 1977, (Presented at Southern Economic Association Meetings beginning 2 November 1977), AD A046 269

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Roberts, Stephen S., "The Decline of the Overseas Station Fleets: The United States Asiatic Fleet and the Shanghai Crisis, 1932," 18 pp., Nov 1977 (Reprinted from The American Neptune, Vol. XXXVII, No. 3, July 1977), AD A047 245

PP 209 – Classified.

PP 210
Kassing, David, "Protecting The Fleet," 40 pp., Dec 1977 (Prepared for the American Enterprise Institute Conference on Problems of Sea Power as We Approach the 21st Century, October 6-7, 1977), AD A049 109

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Mizrahi, Maurice M., "On Approximating the Circular Coverage Function," 14 pp., Feb 1978, AD A054 429

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Mangel, Marc, "On Singular Characteristic Initial Value Problems with Unique Solutions," 20 pp., Jun 1978, AD A058 535

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Mangel, Marc, "Fluctuations in Systems with Multiple Steady States. Application to Lanchester Equations," 12 pp., Feb 78, (Presented at the First Annual Workshop on the Information Linkage Between Applied Mathematics and Industry, Naval PG School, Feb 23-25, 1978), AD A071 472

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Weinland, Robert G., "A Somewhat Different View of The Optimal Naval Posture," 37 pp., Jun 1978 (Presented at the 1976 Convention of the American Political Science Association (APSA/IUS Panel on "Changing Strategic Requirements and Military Posture"), Chicago, Ill., September 2, 1976), AD A056 228

PP 215
Coile, Russell C., "Comments on: Principles of Information Retrieval by Manfred Kochen," 10 pp., Mar 78, (Published as a Letter to the Editor, Journal of Documentation, Vol. 31, No. 4, pages 298-301, December 1975), AD A054 426

PP 216
Coile, Russell C., "Lotka's Frequency Distribution of Scientific Productivity," 18 pp., Feb 1978, (Published in the Journal of the American Society for Information Science, Vol. 28, No. 6, pp. 366-370, November 1977), AD A054 425

PP 217
Coile, Russell C., "Bibliometric Studies of Scientific Productivity," 17 pp., Mar 78, (Presented at the Annual meeting of the American Society for Information Science held in San Francisco, California, October 1976), AD A054 442

PP 218 – Classified.

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Mangel, Marc, "Stochastic Mechanics of Molecule-Ion Molecule Reactions," 21 pp., Jun 1978, AD A056 227

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Mangel, Marc, "Aggregation, Bifurcation, and Extinction In Exploited Animal Populations*," 48 pp., Mar 1978, AD A058 536
*Portions of this work were started at the Institute of Applied Mathematics and Statistics, University of British Columbia, Vancouver, B.C., Canada

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Mangel, Marc, "Oscillations, Fluctuations, and the Hopf Bifurcation*," 43 pp., Jun 1978, AD A058 537
*Portions of this work were completed at the Institute of Applied Mathematics and Statistics, University of British Columbia, Vancouver, Canada.

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Ralston, J. M. and J. W. Mann*, "Temperature and Current Dependence of Degradation in Red-Emitting GaP LEDs," 34 pp., Jun 1978, AD A058 538
*Bell Telephone Laboratories, Inc.

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Mangel, Marc, "Diffusion Theory of Reaction Rates, II Ornstein-Uhlenbeck Approximation," 34 pp., Feb 1978, AD A058 542

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Wilson, Desmond P., Jr., "Naval Projection Forces: The Case for a Responsive MAF," Aug 1978, AD A058 543

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Jacobson, Louis, "An Alternative Explanation of the Cyclical Pattern of Quits," 23 pp., Sep 1978

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Trost, R. P. and Warner, J. T., "The Effects of Military Occupational Training on Civilian Earnings: An Income Selectivity Approach," 38 pp., Nov 1979, AD A077 831

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Powers, Bruce, "Goals of the Center for Naval Analyses," 13 pp., Dec 1978, AD A063 759

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Simpson, William R., "The Accelerometer Methods of Obtaining Aircraft Performance from Flight Test Data (Dynamic Performance Testing)," 403 pp., Jun 1979, AD A075 226

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Weinland, Robert G., "War and Peace in the North: Some Political Implications of the Changing Military Situation in Northern Europe," 18 pp., Nov 1979 (Prepared for presentation to the "Conference of the Nordic Balance in Perspective: The Changing Military and Political Situation," Center for Strategic and International Studies, Georgetown University, Jun 15-16, 1978) AD A077 838

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